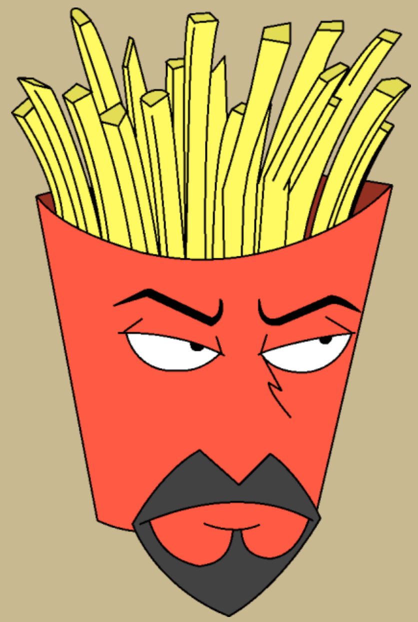
Lecture 1.2

Permutations & combinations



Can I get fries with that?

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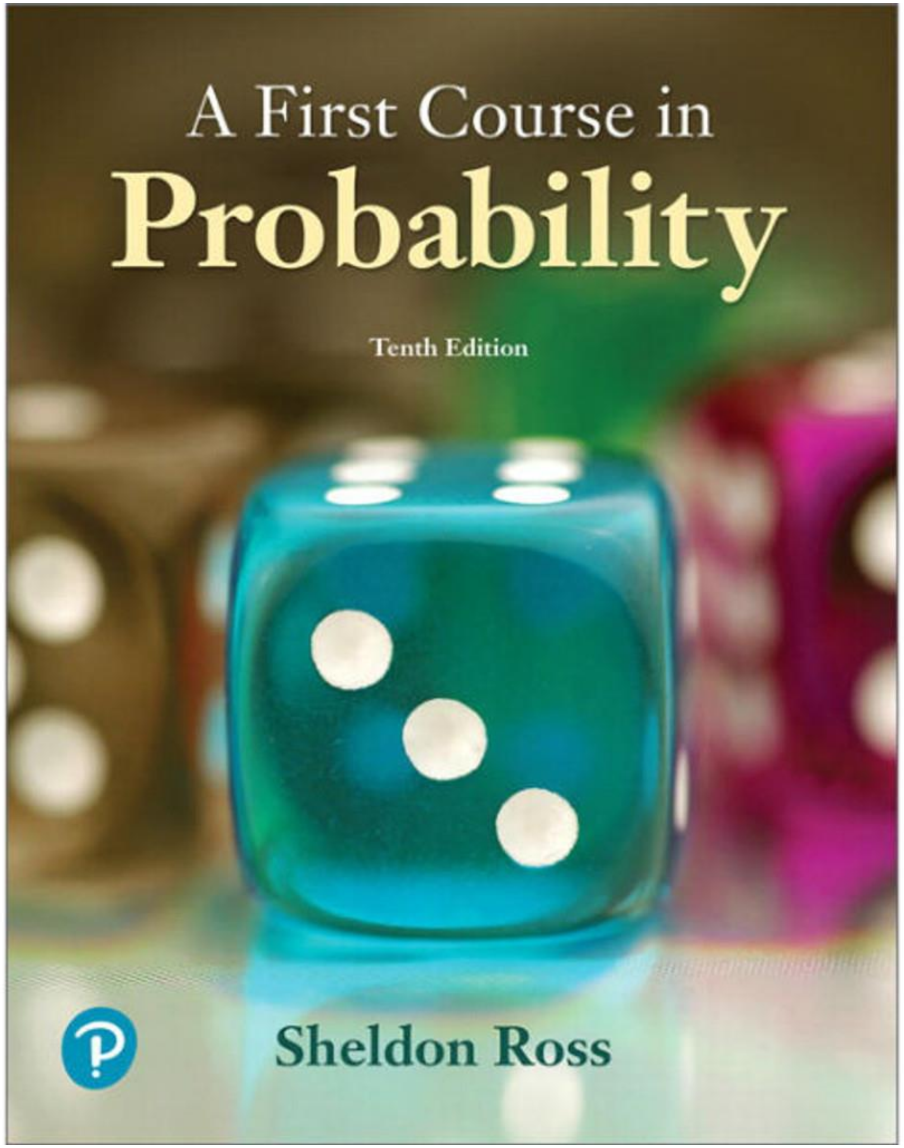


Today's reading: 1.3-1.4

Reading for next class: 1.5

HW1 is now available! Due Friday, 1/24.

Today's draft problem

To be presented by today's draftee.

A standard Indiana license plate always consists of 3 numbers (0-9) followed by 3 letters from the standard Latin/English alphabet (A-Z, capitalized).

(a) How many possible license plates are there?
(b) What if the letters can be in either lower or upper case?
(c) What if no number or letter can be repeated (and we only use upper case)?

Permutations

If we have some things, then the *permutations* of those things are the different ways the things can be arranged in a linear order.

1. If we only had 1 thing, then how many ways can we arrange it? (Not supposed to be a trick question.)
2. If we had 2 things, then how many ways can we arrange them in order? (Also not a trick question)
3. If we had 3 things, then how many ways can we order them?
4. ”Trick” question: how many ways are there to order 0 things?

Answers:
1. 1
2. 2
3. 6
4. 1

Keeping our ducks in a row.

Permutations

In general

If we have n objects, then there are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

ways to permute them. By convention, $0! = 1$.

The ! here is read "factorial."

Permutations with indistinguishable copies?

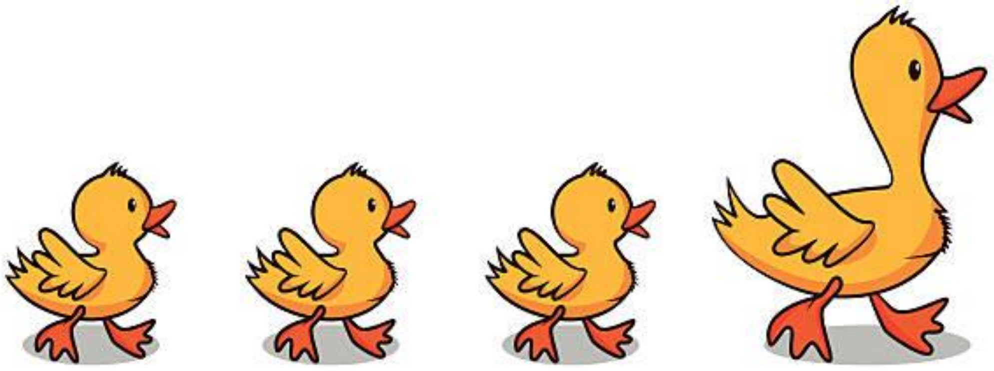
The previous slides assumed the things we had were all distinguishable. However, it also makes sense to consider the permutations of things when we have multiple copies of the same thing.

- How many ways are there to rearrange the letters in the word VIVIENNE?

  Ans: 2520

- What about ANNETTE?

  Ans: 630



What if we can't tell some of our ducks apart?

Permutations with indistinguishable copies

In general

If we have $n = n_1 + n_2 + \cdots n_r$ objects total, of which n_1 are of one type, n_2 are of a second type, …, and n_r are of an r^{th} type, then there are

$$\frac{n!}{n_1!\, n_2! \cdots n_{r!}}$$

ways to permute them.

ANY QUESTIONS ABOUT PERMUTATIONS?

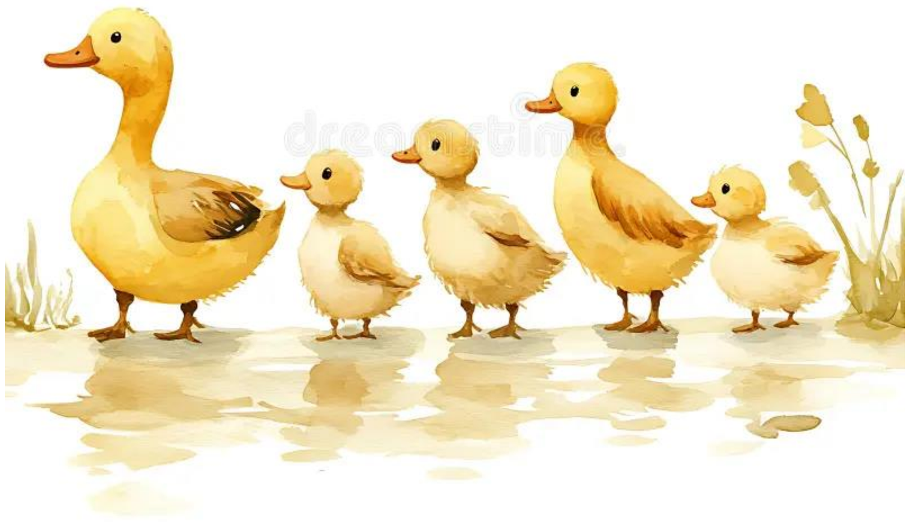
Combinations

If we have n things, then an r *combination* of those things is a way of choosing r of the things (without the order mattering and without repetition).

1. How many ways are there to choose 1 duck out of the 5 on the right?
2. 2 ducks?
3. 3 ducks?
4. 4 ducks?
5. 5 ducks?
6. 0 ducks?
7. 10 ducks?

Answers:

1. 5
2. 10
3. 10
4. 5
5. 1
6. 1
7. 0



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Combinations

In general

If we have n objects total and $0 \leq r \leq n$, then there are

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

ways to choose r of them (without repetition, ignoring the order in which we choose them). If $r < 0$ or $r > n$, then $\binom{n}{r} = 0$.

We read the notation $\binom{n}{r}$ as ”n choose r.”

Pascal's identity

The first of many useful algebraic facts about binomial coefficients.

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Means that when we evaluate this:

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

We get this:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Two proofs of Pascal's identity

"Analytic proof" just means that we prove the identity by doing algebra:

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{((n-1-(r-1))!(r-1)!} + \frac{(n-1)!}{((n-1)-r)!r!} = \cdots$$

$$\cdots = \binom{n}{r} \text{ (see chalkboard)}$$

Two proofs of Pascal's identity

"Combinatorial proof" means that we prove the identity "conceptually:"

Suppose we want to choose r objects out of n objects. We might as well assume that the n objects are "named" $1, 2, \dots, n$. Any time we choose r of them, we either DO choose the object named n or we do NOT choose the object named n. Moreover, these two cases are mutually exclusive. Thus, the number of ways to choose r objects out of n equals the number of ways to choose r objects out of n ASSUMING WE PICK THE OBJECT NAMED n ***PLUS*** the number of ways to choose r objects out of n ASSUMING THAT WE DO NOT PICK THE OBJECT NAMED n.

In the first case, since we are assuming we must choose the object named n, then we are only really choosing $r-1$ objects out of the first $n-1$. In the second case, since we know we're NOT picking the object named n, then we are really just picking r objects out of the first $n-1$ of them.

Thus:

Pick object named n

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Do NOT pick object named n

THAT'S THE HARDEST THING WE'LL DO TODAY

But you DO need to make sure you eventually understand it. It is exactly what "combinatorics" is all about: fancy ways to count things!

Binomial theorem

Another useful algebraic fact about binomial coefficients (and explains their name):

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$$

The book gives two proofs: an “analytic one” (*i.e.* using algebra and mathematical induction) and a “combinatorial one.” You should do your best to understand both.

ANY QUESTIONS ABOUT COMBINATIONS?

Friday's draft problem

To be presented by the draftee determined at the beginning of class on Friday.

How many wraps or bowls can you create at Sushi Boss with the following assumptions:

1. You pick exactly one Protein and exactly 3 Fresh Produce items?
2. Same as before, except you double your protein?
3. You pick either 0 or 1 Proteins and 0, 1, 2 or 3 Fresh Produce items?

LET'S DO SOME PRACTICE PROBLEMS NOW