Lecture 1.3

Multinomial coefficients

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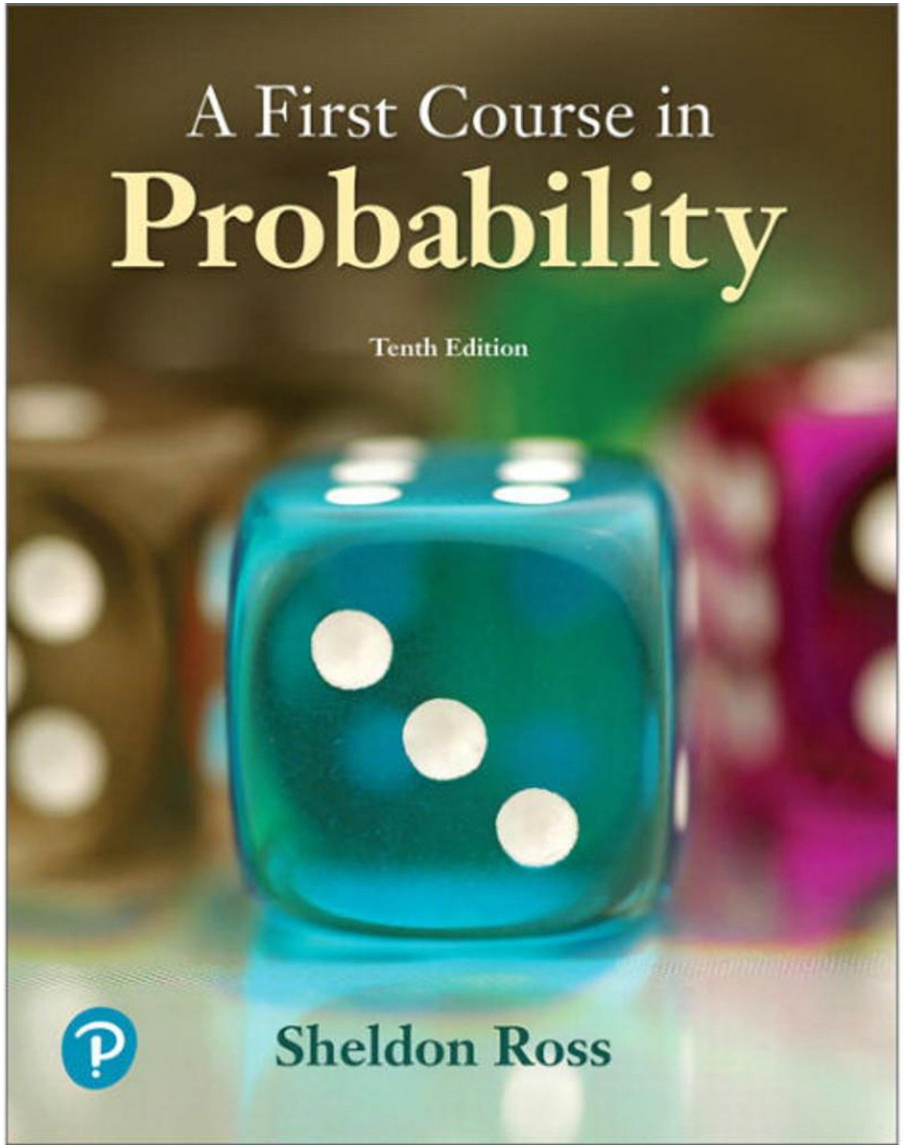


Today's reading: 1.5

Reading for next class: 2.1 + 2.2

Don't forget: no class Monday due to MLK Day holiday. (No office hour either.)

Today’s draft problem

To be presented by today’s draftee.

How many wraps or bowls can you create at Sushi Boss with the following assumptions:

1. You pick exactly one Protein and exactly 3 Fresh Produce items?
2. Same as before, except you double your protein?
3. You pick either 0 or 1 Proteins and 0, 1, 2 or 3 Fresh Produce items?

Putting things in buckets

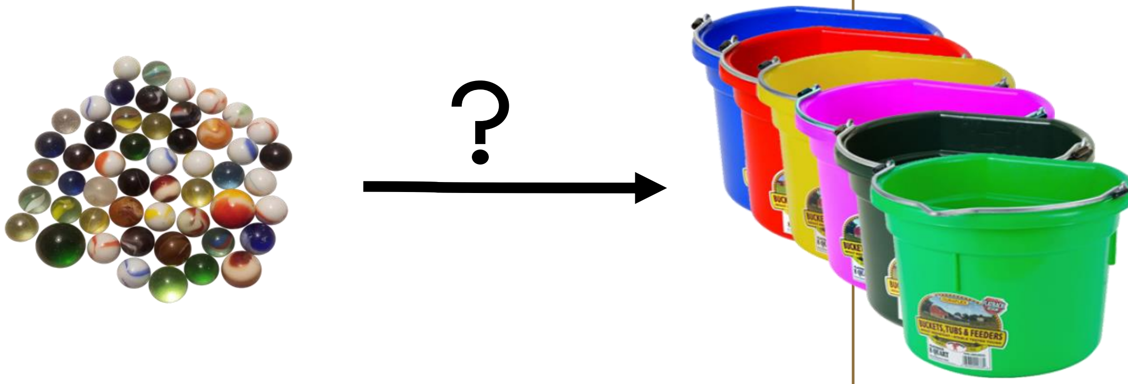
How many ways are there to put M (distinct) marbles into B (distinct) buckets? (Not a trick question)

?

Answer: B^M

Putting specific numbers of things in buckets?

How many ways are there to put M (distinct) marbles into B (distinct) buckets if we are required to put m_1 marbles in the first bucket, m_2 marbles in the second bucket, …, m_B in the last bucket?



Answer: It depends.

If $m_1 + m_2 + \cdots + m_b > M$, then the answer is 0.

So let's assume $m_1 + m_2 + \cdots + m_b \leq M$ and figure it out.

Figuring it out

How many ways are there to put M (distinct) marbles into B (distinct) buckets if we are required to put m_1 marbles in the first bucket, m_2 marbles in the second bucket, …, m_B in the last bucket, assuming $m_1 + m_2 + \cdots + m_b \leq M$?

Answer: We might as well work through the buckets one-by-one. For the first bucket, we can choose any m_1 marbles from the M total we have. After making that choice, for the second bucket, we can choose any m_2 marbles from the remaining $M - m_1$ we have. For the third bucket, we can choose any m_3 marbles from the remaining $M - m_1 - m_2$ we have. Continuing in this way, we see that we have

$$\binom{M}{m_1}\binom{M-m_1}{m_2}\cdots\binom{M-m_1-m_2-\cdots-m_{b-1}}{m_b}$$

many ways total.

Simplifying

How many ways are there to put M (distinct) marbles into B (distinct) buckets if we are required to put m_1 marbles in the first bucket, m_2 marbles in the second bucket, …, m_B in the last bucket, assuming $m_1 + m_2 + \cdots + m_b \leq M$?

$$\binom{M}{m_1}\binom{M-m_1}{m_2}\binom{M-m_1-m_2}{m_3}\cdots\binom{M-m_1-m_2-\cdots-m_{b-1}}{m_b}$$

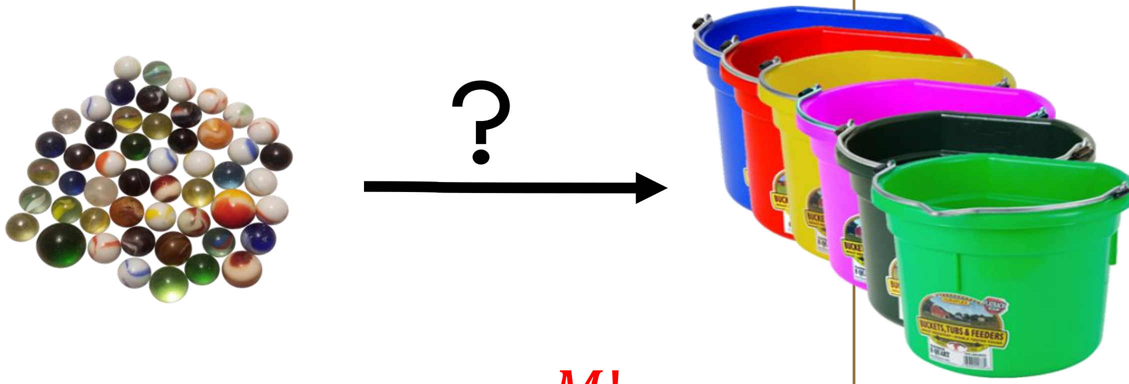
$$= \frac{M!}{m_1!\,(M-m_1)!}\frac{(M-m_1)!}{m_2!\,(M-m_1-m_2)!}\frac{(M-m_1-m_2)!}{m_3!\,(M-m_1-m_2-m_3)!}\cdots\frac{(M-m_1-m_2-\cdots-m_{b-1})!}{m_b!\,(M-m_1-m_2-\cdots-m_{b-1}-m_b)!}$$

$$= \frac{M!}{m_1!\,m_2!\cdots m_b!\,(M-m_1-m_2-\cdots-m_{b-1}-m_b)!}$$

Important special case

How many ways are there to put M (distinct) marbles into B (distinct) buckets if we are required to put m_1 marbles in the first bucket, m_2 marbles in the second bucket, …, m_B in the last bucket, assuming every marble goes into a bucket—that is, assuming $m_1 + m_2 + \cdots + m_b = M$?

Answer: The work we just did simplifies.



$$\frac{M!}{m_1!\, m_2! \cdots m_b!\, (M - m_1 - m_2 - \cdots - m_{b-1} - m_b)!} = \frac{M!}{m_1!\, m_2! \cdots m_b!\, 0!}$$

$$= \frac{M!}{m_1!\, m_2! \cdots m_b!}$$

Multinomial coefficients

Definition:

If $n_1 + n_2 + \cdots + n_r = n$, we define the *multinomial coefficient* $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

The number $\binom{n}{n_1, n_2, \ldots, n_r}$ counts the number of ways to divide n distinct things into r distinct groups of sizes $n_1, n_2, \ldots$,and n_r (respectively).

An example with two approaches

Suppose we play a card game with a standard deck of 52 cards and 4 players (distinct buckets) where every player gets dealt 10 cards. How many deals are possible?

Solution 1: Have 52 “marbles” (cards) and 4 “buckets” (players), but not every marble will go in a bucket.

Using our analysis from above, the number of possible deals is:

$$\binom{52}{10}\binom{42}{10}\binom{32}{10}\binom{22}{10}$$

An example with two approaches

Suppose we play a card game with a standard deck of 52 cards and 4 players (distinct buckets) where every player gets dealt 10 cards. How many deals are possible?

Solution 2: Have 52 “marbles” (cards) and 5 “buckets” (4 players, along with a fifth special bucket that will hold the cards that don’t get dealt). 10 cards will go into each of the player buckets, and 12 cards will go into the special bucket.

Thus, the number of possible deals is:

$$\binom{52}{10, 10, 10, 12}$$

You should be sure you grok why these two approaches give the same answer!!

Something fun to think about

We saw on Wednesday that if we have a word of length n consisting of r types of letters, where the i^{th} letter is repeated n_i times (so $n_1 + n_2 + \cdots + n_r = n$), then there are

$$\frac{n!}{n_1!\,n_2!\cdots n_r!} = \binom{n}{n_1, n_2, \dots, n_r}$$

many ways to permute the letters of the word.

Can you see for yourself why this "should" be a multinomial coefficient?

(Hint: think of the *types* of letters as the buckets, and the different *positions* of the letters in the words as the marbles.)

Multinomial theorem

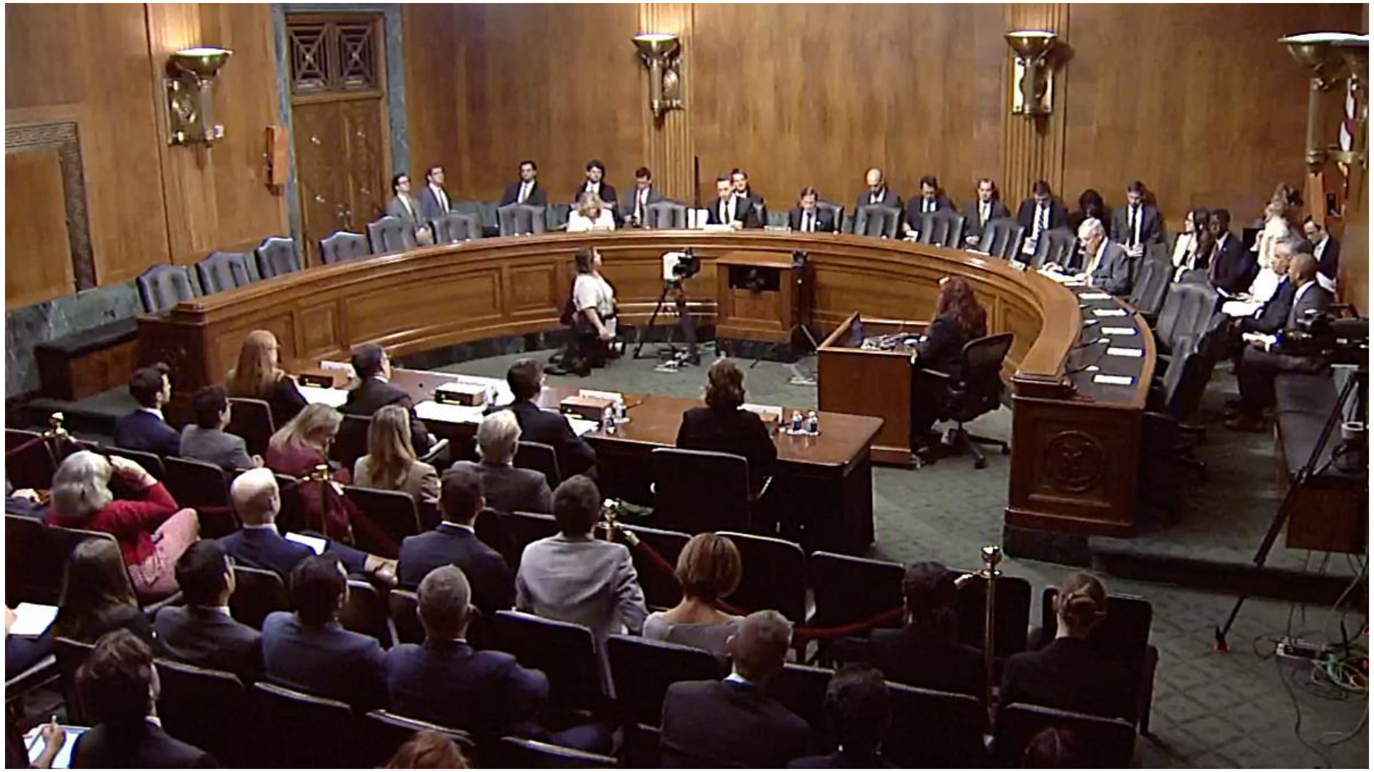
Generalizes binomial theorem:

$$\boxed{(x_1 + x_2 + \cdots x_r)^n = \sum_{\substack{(n_1, n_2, \ldots, n_r)\,: \\ n_1 + n_2 + \cdots + \mathrm{n}_r = n}} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}}$$

ANY QUESTIONS ABOUT MULTINOMIAL COEFFICIENTS?

Wednesday's draft problem

To be presented by the draftee at the beginning of class on Wednesday.



a) Give an algebraic proof that

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}.$$

a) Now give a combinatorial proof of the same identity. (Hint: the picture on the left is of a Congressional subcommittee.)

LET'S DO SOME PRACTICE EXERCISES NOW