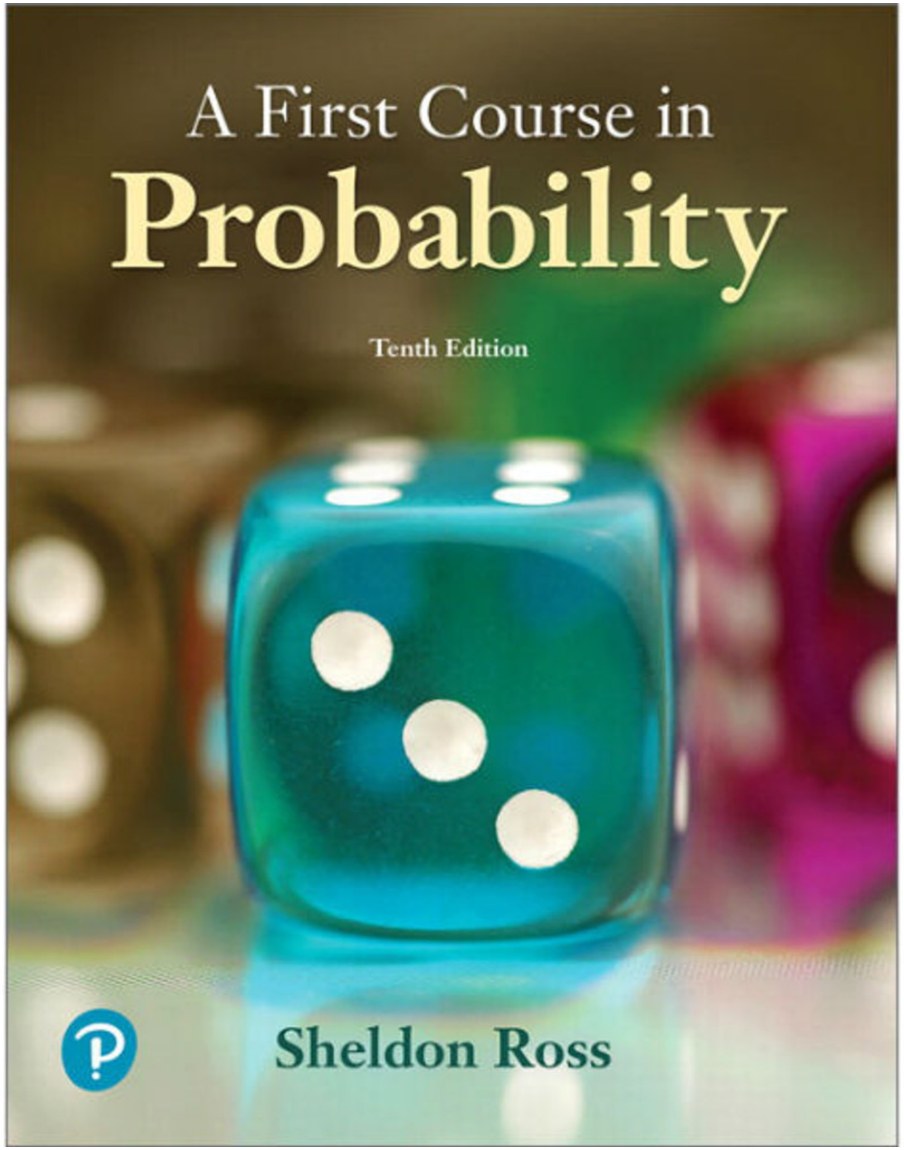
Lecture 10.1

Exponential random variables

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 5.5

Next class: 4.8.1

HW8 due Friday.

Today’s draft problem

To be presented by today’s draftee

Variation of Ex. 4h

The 10 year average for Purdue’s undergraduate yield rate (that is, the fraction of admitted undergrad applicants who accept their admissions offer and show up) is 25.1%. Assuming that the admissions office’s goal was to have 9,815 freshman show up in fall 2024, they made 39,106 acceptances.

Use a normal approximation to determine the probability that 1,600 or more students than the goal show up.

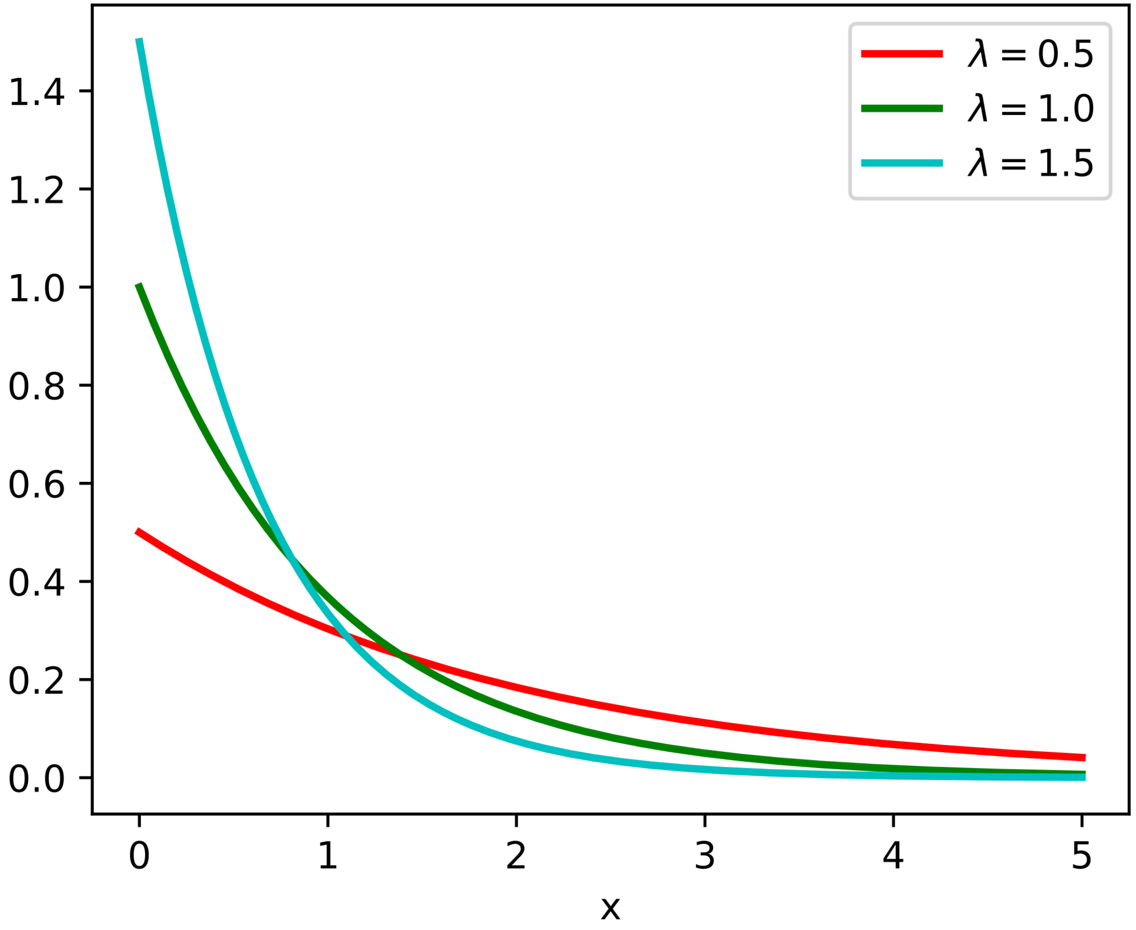
Exponential Random Variables

Definition:

A continuous $\mathbb{R}$-valued random variable X (on a sample space S with probability measure P…) is called *exponential* with parameter (or rate) λ if its PDF is given by

$$f(x) = f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Exponential Random Variables



Exponential Random Variables – CDF, expectation and variance

Let X be exponentially distributed with parameter (or rate) λ. Then:

- the CDF of X is $F(a) = \begin{cases} 1 - e^{-\lambda a}, & a \geq 0 \\ 0, & a < 0 \end{cases}$
- $E[X] = 1/\lambda$
- $Var(X) = 1/\lambda^2$

Memorylessness

Definition:

A (nonnegative) random variable X is called *memoryless* if

$$P\{X > s + t \mid X > t\} = P\{X > s\}$$

for all $s, t \geq 0$.

Equivalently (why?):

$$P\{X > s + t\} = P\{X > s\}P\{X > t\}$$

Memoryless ⇔ Exponential

Fact:

A (nonnegative and continuous) random variable X is memoryless if and only if it is exponential.

⇐: easier. Let's show it now.

⇒: takes more work. We'll prove it momentarily using "hazard rate functions."

A brain teaser

Example 5c Consider a post office that is staffed by two clerks. Suppose that when Mr. Smith enters the system, he discovers that Ms. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Ms. Jones or Mr. Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter λ, what is the probability that of the three customers, Mr. Smith is the last to leave the post office?

Hazard rate functions

Definition:

Let X be any positive, continuous random variable with PDF f and CDF F. Then the *hazard rate function* (or *failure rate function*) of X is the function

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

Hazard rate functions - interpretation

Interpretation:

If X represents the time of failure of some Thing, then its hazard rate function (aka failure rate function) represents the "conditional probability intensity" of the Thing failing "very soon," given that it has survived for t time.

Somewhat more precisely:

$$P\{X \in (t, t + dt) | X > t\} = \frac{P\{X \in (t, t + dt), X > t\}}{P\{X > t\}} = \frac{P\{X \in (t, t + dt)\}}{P\{X > t\}}$$

$$\approx \frac{f(t)\, dt}{1 - F(t)} = \lambda(t) dt$$

Hazard rate function of exponential random variables

What is the hazard rate function of an exponentially distributed random variable?

Let X be exponentially distributed with parameter (or rate) λ. Then its hazard rate function is constant:

$$\lambda(t) = \lambda$$

Hazard rate functions can “do it all”

Fact:

The hazard rate function of a (positive and continuous) random variable can completely recover the random variable’s CDF.

Why?

Hint: consider

$$\int_0^t \lambda(s)\, ds$$

Memoryless ⇒ Exponential

Let's finish the proof that "memoryless" and "exponential" are equivalent:

- If X is memoryless, then one can show that its hazard rate function is constant, *i.e.*

$$\lambda(s) = c$$

(How?)

- We just showed that

$$F(t) = 1 - exp\left\{-\int_0^t \lambda(s)\, ds\right\}$$

- Thus: $F(t) = 1 - e^{-ct}$, and so $f(t) = \frac{dF}{dx}(t) = ce^{-ct}$, which shows the variable is exponential.

Wednesday's draft problem

To be presented by Wednesday's draftee.

Let X be an exponentially distributed random variable with rate λ. Compute the n^{th} moment of X, *i.e.* $E[X^n]$.

(Hint: use induction.)