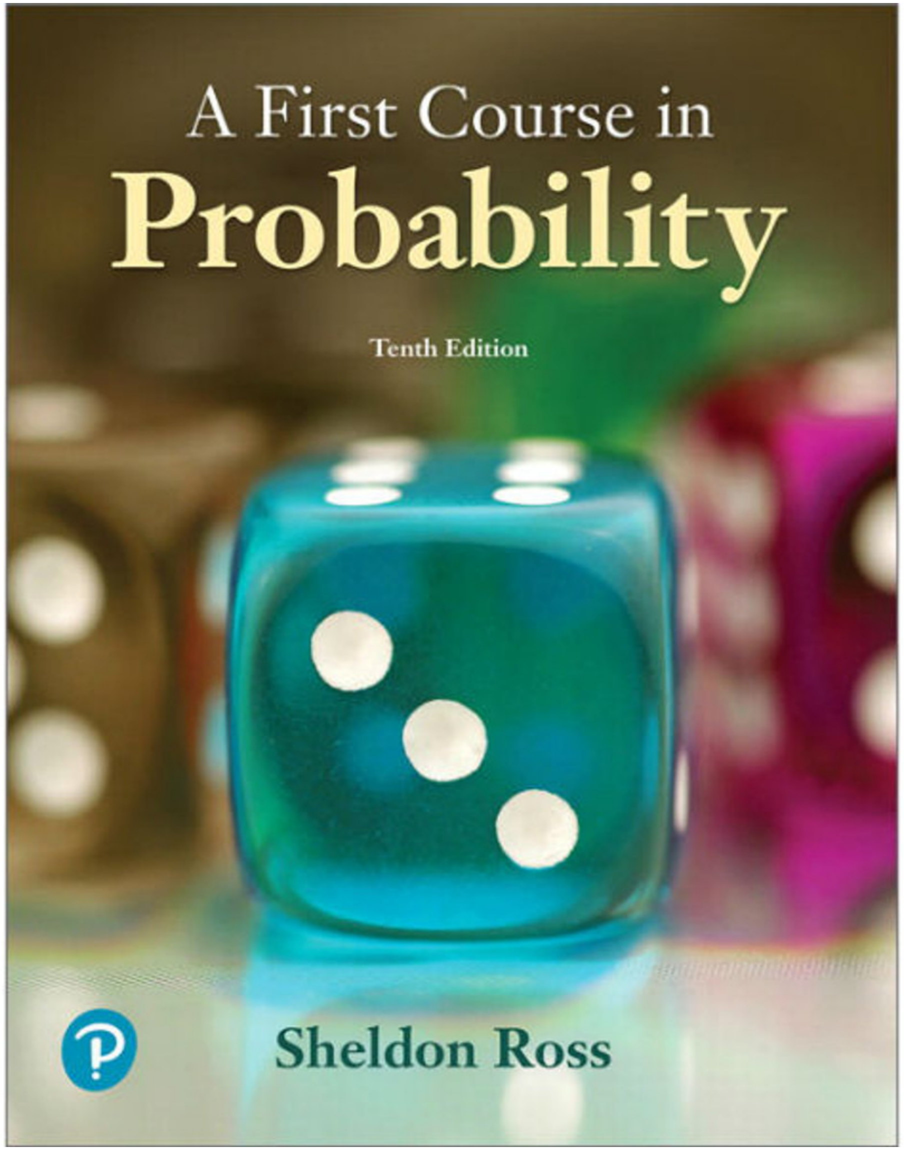
Lecture 10.2

More on memorylessness; geometric random variables

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: more from 5.5, 4.8.1

Next class: 5.7

HW8 due Friday.

Department of Mathematics

Today’s draft problem

To be presented by today’s draftee.

Let X be an exponentially distributed random variable with rate λ. Compute the n^{th} moment of X, *i.e.* $E[X^n]$.

(Hint: use induction.)

Recall: Exponential Random Variables

Definition:

A continuous $\mathbb{R}$-valued random variable X (on a sample space S with probability measure P...) is called *exponential* with parameter (or rate) λ if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Basic properties (proved last class):

- the CDF of X is $F(a) = \begin{cases} 1 - e^{-\lambda a}, & a \geq 0 \\ 0, & a < 0 \end{cases}$
- $E[X] = 1/\lambda$
- $Var(X) = 1/\lambda^2$

Recall: Hazard rate functions

Definition:

Let X be any positive, *continuous* random variable with PDF f and CDF F. Then the *hazard rate function* (or *failure rate function*) of X is the function

$$\lambda(t) = \frac{f(t)}{1 - F(t)}$$

Hazard rate functions can “do it all”

Fact:

The hazard rate function of a (*positive* and continuous) random variable can completely recover the random variable’s CDF.

Why?
Hint: consider

$$\int_0^t \lambda(s)\, ds$$

Final answer: $F(t) = 1 - exp\left\{-\int_0^t \lambda(s)\, ds\right\}$

Hazard rate function of exponential random variables

What is the hazard rate function of an exponentially distributed random variable?

Let X be exponentially distributed with parameter (or rate) λ. Then its hazard rate function is constant:

$$\lambda(t) = \lambda$$

Recall: Memorylessness

Definition:

A (nonnegative) random variable X is called *memoryless* if

$$P\{X > s + t \mid X > t\} = P\{X > s\}$$

for all $s, t \geq 0$. Equivalently:

$$P\{X > s + t\} = P\{X > s\}P\{X > t\}$$

Fact:

A nonnegative, ***continuous*** random variable X is memoryless if and only if it is exponential.

We didn't finish proving the ⇒ direction, so let's do it now.

Continuous + Memoryless ⇒ Exponential

Let's finish the proof that "continuous + memoryless" and "exponential" are equivalent:

- If X is memoryless and continuous, then one can show that its hazard rate function is constant, *i.e.*

$$\lambda(s) = c$$

(How?)

- We just showed that

$$F(t) = 1 - exp\left\{-\int_0^t \lambda(s)\, ds\right\}$$

- Thus: $F(t) = 1 - e^{-ct}$, and so $f(t) = \frac{dF}{dx}(t) = ce^{-ct}$, which shows the variable is exponential.

So, exponential random variables are exactly the same thing as continuous random variables that are memoryless.

But what about *discrete* random variables? Which of those are memoryless?

Geometric random variables

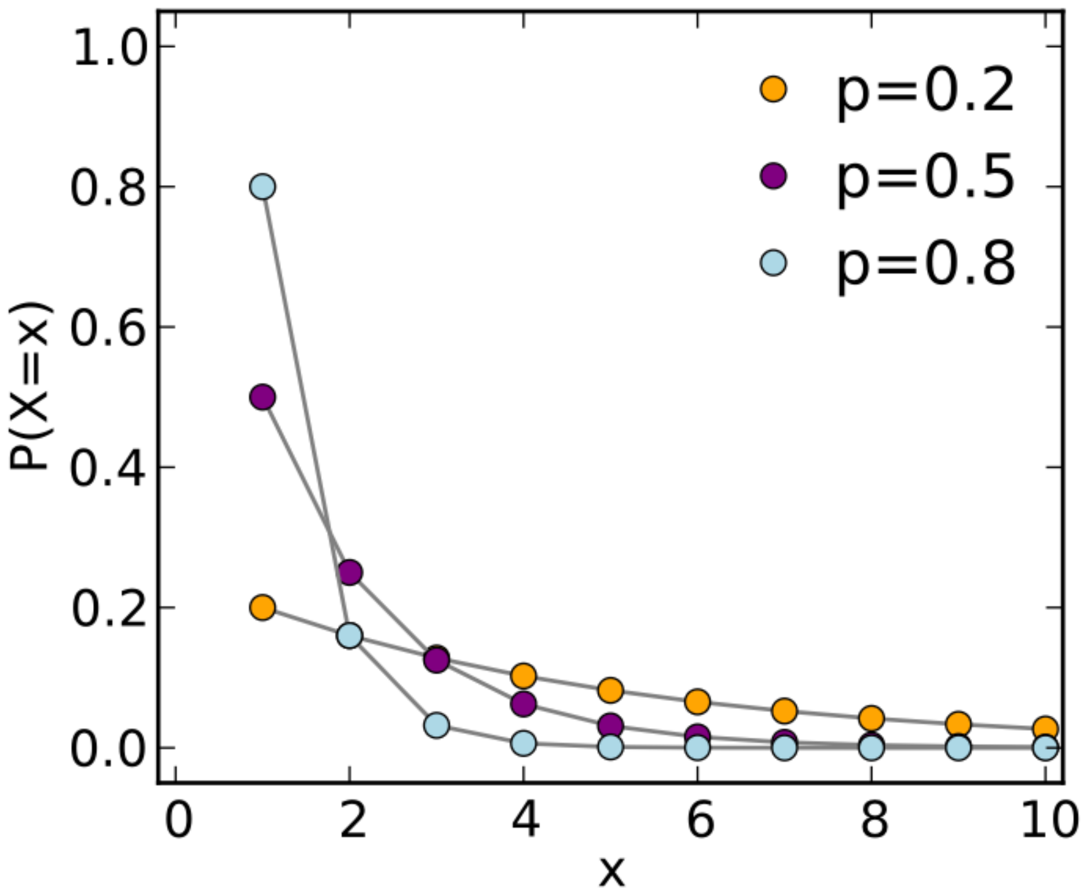
Definition:

A *discrete* $\mathbb{R}$-valued random variable X (on a sample space S with probability measure P...) is called <u>*geometric*</u> with success probability $0 \leq p \leq 1$ if its PMF is given by

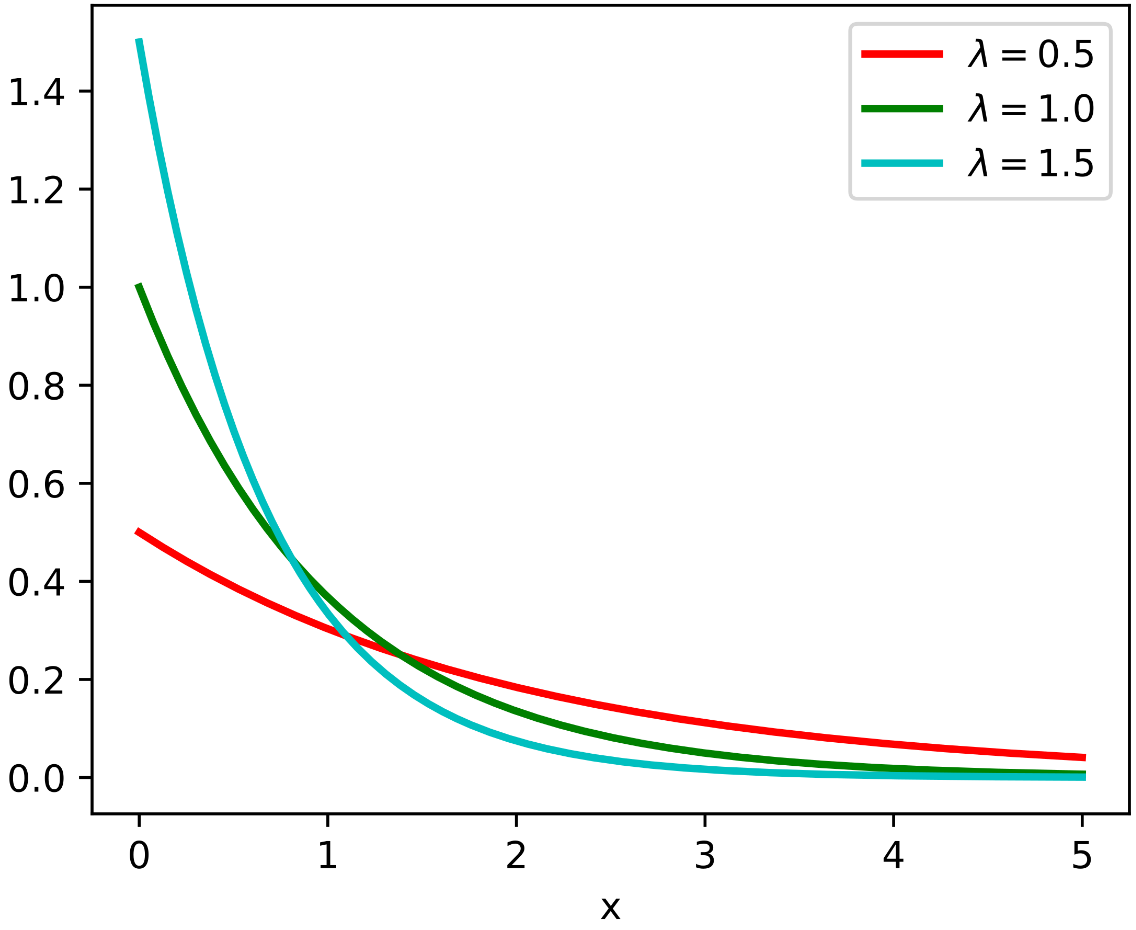
$$P\{X = n\} = (1-p)^{n-1}p \qquad n = 1,2,3,\ldots$$

Equivalently: a geometric random variable keeps track of ***when we see the first success*** if we repeatedly do independent Bernoulli trials, each with success probability p.

Geometric random variables



Recall: Exponential Random Variables



Geometric random variables – basic properties

If X is a geometric random variable with success probability p then:

- $E[X] = 1/p$
- $Var(X) = \frac{1-p}{p^2}$

Read Examples 8b and 8c in Section 4.8.1 to see why.

Discrete memorylessness

Fact:

A positive, ***discrete*** random variable X is memoryless if and only if it is geometric.

Let's prove the easy direction, namely: $\Leftarrow$

Note how intuitive this is, e.g.: if I repeatedly flip a coin until I get the first heads, then the probability that it requires at least 50+30 flips given that I've already gotten 30 tails in a row IS EXACTLY EQUAL to the probability that it takes at least 50 flips. ***The coin does not remember that it had 30 tails in a row!***

We won't prove the harder $\Rightarrow$ direction. (Note that we can't use hazard rate functions because we're in the discrete setting! Need to solve a recurrence relation instead.)

Exponential random variables are a continuous limit of geometric random variables

Fact:

If $p = \frac{\lambda}{n}$ and X is geometric with success probability p, then as $n \to \infty$ the CDF of X converges to an exponential random variable with parameter λ.

Analogy:

Exponential random variables are related to geometric random variables like Poisson random variables are related to binomial random variables.

Friday's draft problem

To be presented by Friday's draftee

Let X be a geometric random variable. Give both an algebraic proof and a verbal explanation of the following:

$$P\{X = n + k \mid X > n\} = P\{X = k\}$$