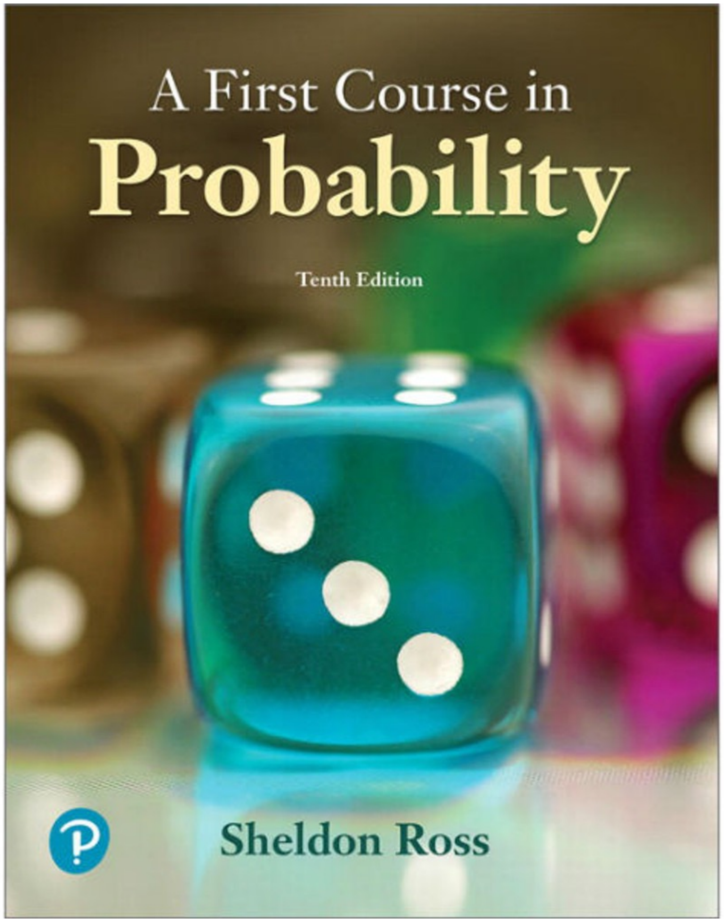
Lecture 11.2

Independent random variables

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: more from 6.1, 6.2 (Not on MT2)

Next class: MT2 Review

HW9 now available. Due Friday.

Practice MT2 is now available. Wil send more study suggestions later today.

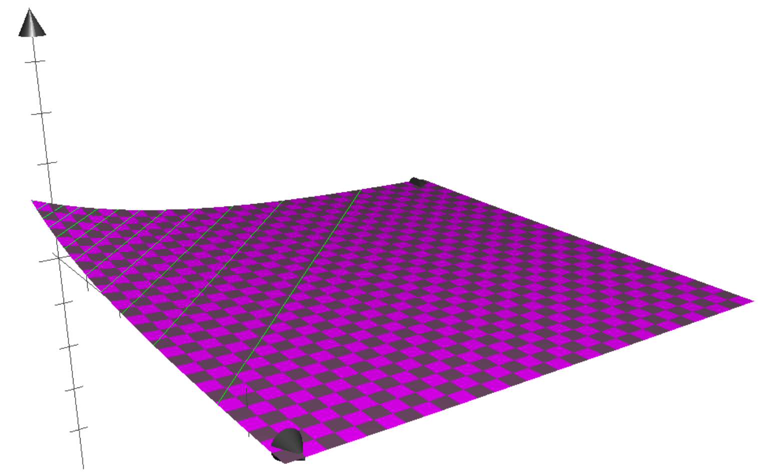
Today's draft problem

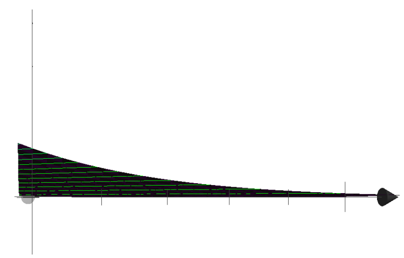
To be presented by today's draftee.

Suppose X and Y are jointly continuously distributed with joint PDF

$$f(x,y) = \begin{cases} e^{-(x+y)}, & x > 0, y > 0 \\ 0, & \text{else.} \end{cases}$$

Find $P\{X < Y\}$ and $P\{X < a\}$.





Recall: Jointly continuous random variables

Definition:

Let X and Y be random variables. We say they are ***jointly continuous*** if there exists a function $f(x, y)$ such that for every (measurable) event $C \subset \mathbb{R}^2$

$$P\{(X, Y) \in C\} = \iint\limits_{(x,y)\in C} f(x, y)\, dx\, dy$$

We call f the *joint probability density function* (*joint PDF*) of X and Y.
In particular,

$$F(a, b) = \int_{-\infty}^{b} \int_{-\infty}^{a} f(x, y)\, dx\, dy$$

so by the Fundamental Theorem of Calculus,

$$f(a, b) = \frac{\partial^2}{\partial a\, \partial b} \mathrm{F(a, b)}$$

Recall: Jointly continuous random variables

We recover marginals in the continuous case analogously to how we did it in the discrete case:

Let X and Y be jointly continuous random variables. Then

$$P\{X \in A\} = P\{X \in A, -\infty < Y < +\infty\} = \int_A \int_{-\infty}^{+\infty} f(x,y)\, dy\, dx = \int_A f_X(x)\, dx$$

where

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\, dy$$

is the PDF of X.

Let's do examples 1f and 1e now.

SEVERAL jointly random variables

We can generalize all the above from two random variables to several.

Let $X_1, X_2, \dots, X_n$ be several random variables. Their *joint CDF* is defined by

$$F(a_1, a_2, \dots, a_n) = P\{X_1 \le a_1, X_2 \le a_2 \dots, X_n \le a_n\}.$$

We say they are *jointly continuous* if there exists a function $f(x_1, x_2, \dots, x_n)$ such that

$$P\{(X_1, X_2, \dots, X_n) \in C\} = \iint \cdots \int_{(x_1, x_2, \dots, x_n) \in C} f(x_1, x_2, \dots, x_n)\, dx_1 \cdots dx_n$$

We recover the marginal for, say, X_1 by "integrating out" all the other variables:

$$P(X_1 \in A_1) = \int_{A_1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, x_2, \dots, x_n)\, dx_2 \cdots dx_{n-1}\, dx_n$$

Independent random variables

Definition:

Two random variables X and Y are independent if for any two (measurable*) subsets of real numbers $A, B \subset \mathbb{R}$

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}.$$

In other words: the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for all $A, B \subset \mathbb{R}$.

With some work, previous definition can be shown to be equivalent to following:

$$F(a,b) = F_X(a)F_Y(b) \qquad \text{for all } a, b \in \mathbb{R}.$$

Discrete Case: Independent random variables

Useful fact:

If two random variables X and Y are both discrete, then they are independent exactly when

$$p(x, y) = p_X(x)p_Y(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

Let's prove this.

Continuous Case: Independent random variables

Useful fact:

If two random variables X and Y are both continuous, then they are independent exactly when

$$f(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}.$$