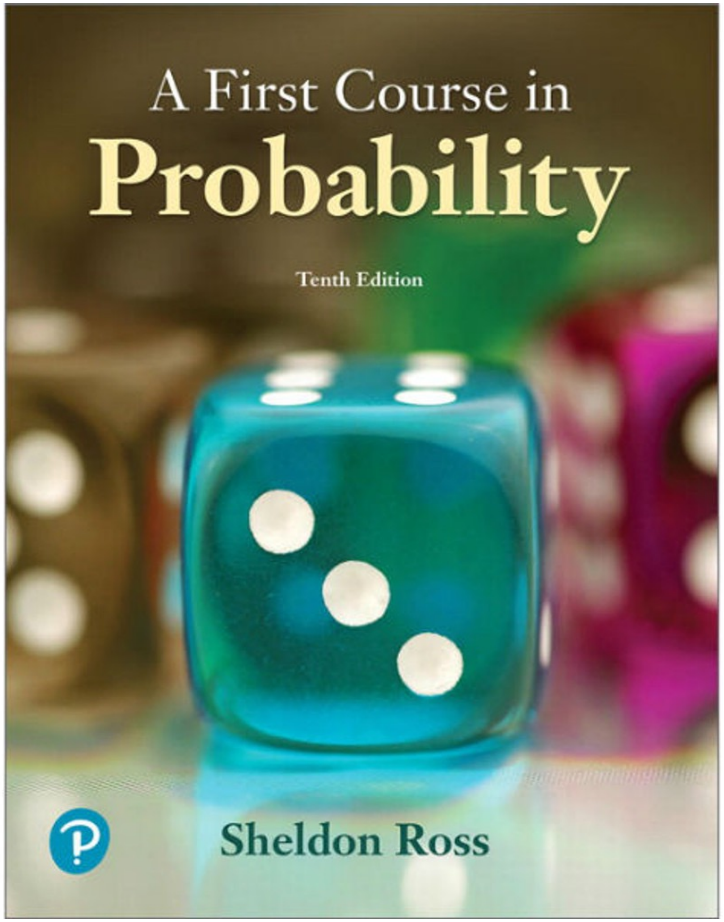
Lecture 12.3

Sums of Independent Random Variables

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 6.3

Next class: 6.4

HW10 now available. Due next Friday, 4/18.

I will do my best to get graded MT2 back to you by Monday.

I will *definitely* get it back to you before next Friday, since that is the last day to drop (with W).

Remember: starting this Friday (4/11), we will have TWO draft problems every day, an "A" problem and a "B" problem. If you will re-announce the A draftee lists and B draftee lists now.

Today's draft problem A

To be presented by today's A draftee.

Use Chebyshev's inequality to answer the following:

How many flips n of a biased coin with unknown probability of heads p does it take in order to be 90% certain that the ratio

$$\frac{\text{number of heads observed}}{n}$$

agrees with p to two decimal places?

Put another way: if $0 \leq p \leq 1$ and X_n denotes a binomial random variable with parameters n and p where p is unknown, how large does n need to be in order to guarantee that

$$P\left\{\left|\frac{X_n}{n} - p\right| \geq 0.01\right\} \leq 0.1$$

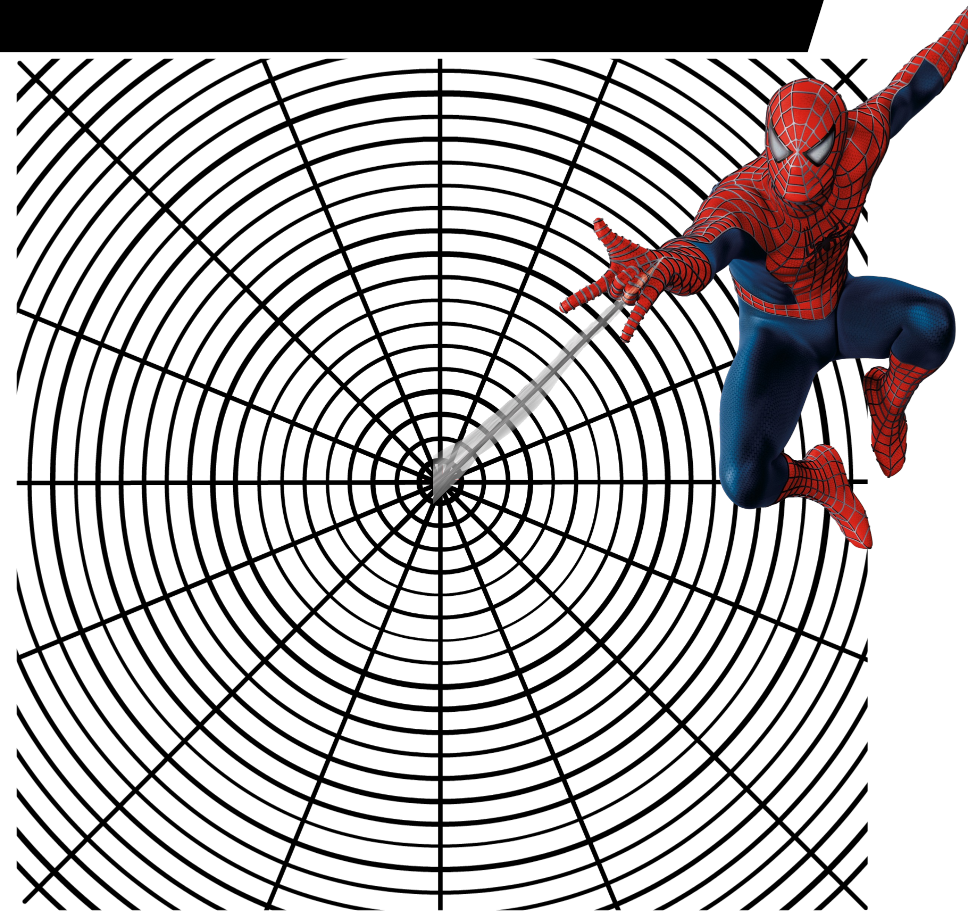
Hint: $\left|\frac{X_n}{n} - p\right| \geq 0.01$ if and only if $|X_n - \mu_n| \geq 0.01n$, where $\mu_n = np = E[X_n]$. Now combine Chebyshev's inequality with the fact that $\sigma_n^2 = Var(X_n) \leq 0.25n$ (the key point is that this is independent of p).

Today's draft problem B

To be presented by today's B draftee.

Consider two independent random variables X_1 and X_2 such that X_1 follows an exponential distribution with mean 2, and X_2 follows a uniform distribution on the interval $[0,2\pi]$. Let $Y_1 = \sqrt{X_1}\cos X_2$ and $Y_2 = \sqrt{X_1}\sin X_2$.

1. Show that Y_1 and Y_2 are independent.
2. Show that Y_1 and Y_2 are both standard normal variables.



Question of the day

We're going to answer the following question for several examples today:

If X and Y are independent random variables, then what is the CDF/PDF/PMF of $X + Y$?

(Of course, this question is also interesting when X and Y are not independent, but it's quite hard then!)

Question of the day

We're going to answer the following question for several examples today:

We can give a general "formal" answer in the case that X and Y are both continuous (however, this "formal" answer still leaves it up to us to have to solve some integrals…)

$$F_{X+Y}(a) = P\{X + Y \le a\} = \iint\limits_{x+y\le a} f_X(x) f_Y(y) dx\, dx = \int_{-\infty}^{+\infty} \int_{-\infty}^{a-y} f_X(x) f_Y(y) dx\, dy = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{a-y} f_X(x) dx \right) f_Y(y) dy$$

$$= \int_{-\infty}^{+\infty} F_X(a-y) f_Y(y)\, dy$$

$$f_{X+Y}(a) = \frac{d}{da} \int_{-\infty}^{+\infty} F_X(a-y) f_Y(y)\, dy = \int_{-\infty}^{+\infty} \frac{d}{da} F_X(a-y) f_Y(y)\, dy = \int_{-\infty}^{+\infty} f_X(a-y) f_Y(y) dy$$

"convolution"

Example 3a: X and Y both uniform on (0,1)

Let's find the PDF of $X + Y$ on the chalkboard.

Answer:

$$f_{X+Y}(a) = \begin{cases} a & if\ 0 \leq a \leq 1 \\ 2 - a & if\ 1 < a < 2 \\ 0 & else \end{cases}$$

Sums of normal random variables

Let X and Y be independent normal random variables with parameters (μ_X, σ_X^2) and (μ_Y, σ_Y^2), respectively. Let's figure out what $X + Y$ looks like.

Hint: if X and Y are any independent random variables, then what are $E[X + Y]$ and $Var(X + Y)$?

Answer: $X + Y$ is normal with parameters $\mu_X + \mu_Y$ and $\sigma_X^2 + \sigma_Y^2$.

More generally, by induction, we get:

Proposition 3.2

If X_i, $i = 1, \dots, n$, are independent random variables where each is normal with parameters $\mu_i, \sigma_i^2, i = 1, \dots, n$, then $\sum_{i=1}^{n} X_i$ is normally distributed with parameters $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$.

Discrete example: Sums of Poisson random variables

Let X and Y be independent Poisson random variables with parameters λ_X and λ_Y, respectively. Let's figure out what $X + Y$ looks like on the chalk board.

Answer: $X + Y$ is Poisson with parameter $\lambda_X + \lambda_Y$.

Monday's draft problem A

To be presented by Monday's A draftee.

At Dr. Gesund's office, the waiting time T is modeled by an exponential random variable with mean 10 minutes. Today the office proposes the following deal: if your waiting time is less than 20 minutes, then you pay the full amount of your visit. Otherwise, you get reimbursed your waiting time minus 20. We call X the amount which is reimbursed by the office. Find the CDF of X. Then find the probability that you get reimbursed twice in 5 visits.

Friday's draft problem B

To be presented by Friday's B draftee.

Let X and Y be independent binomial random variables with parameters (n, p) and (m, q), respectively. Compute the PMF of $X + Y$.