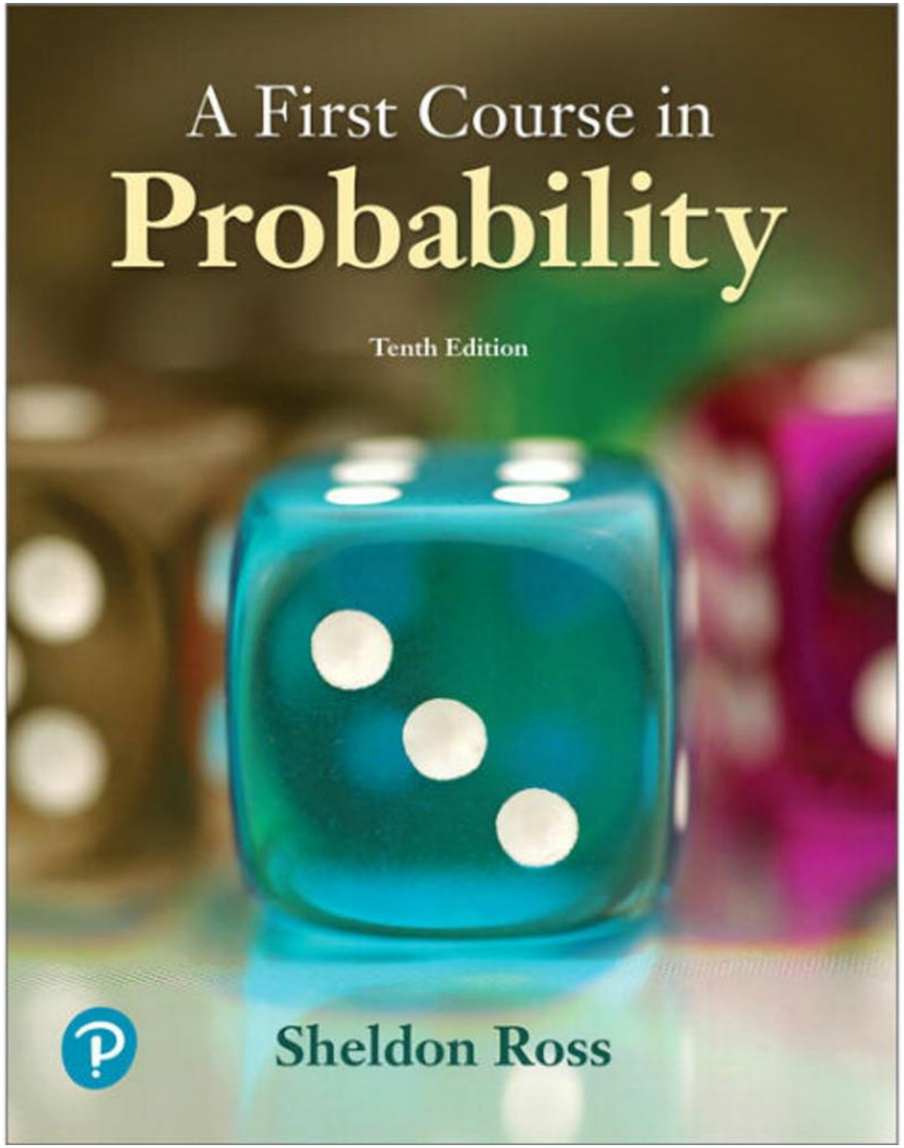
Lecture 13.2

Conditional distributions: continuous case

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 6.5

Next class: 7.1+7.2

HW10 due Friday.

HW11 will be your LAST homework assignment! Available now, due 4/25

MT2 has been graded. Stats on next slides.

Rough letter grade estimate for current Final Calculated Grades:

A range: 56-65

B range: 50-56

C range: 40-50

D range: 30-40

F range: less than 30

DROP DAY IS TOMORROW, THURSDAY, 4/17!

Midterm Exam 2 Class Statistics

Number of submitted grades: 84 / 84

Minimum: 0 %

Maximum: 100 %

Average: 73.21 %

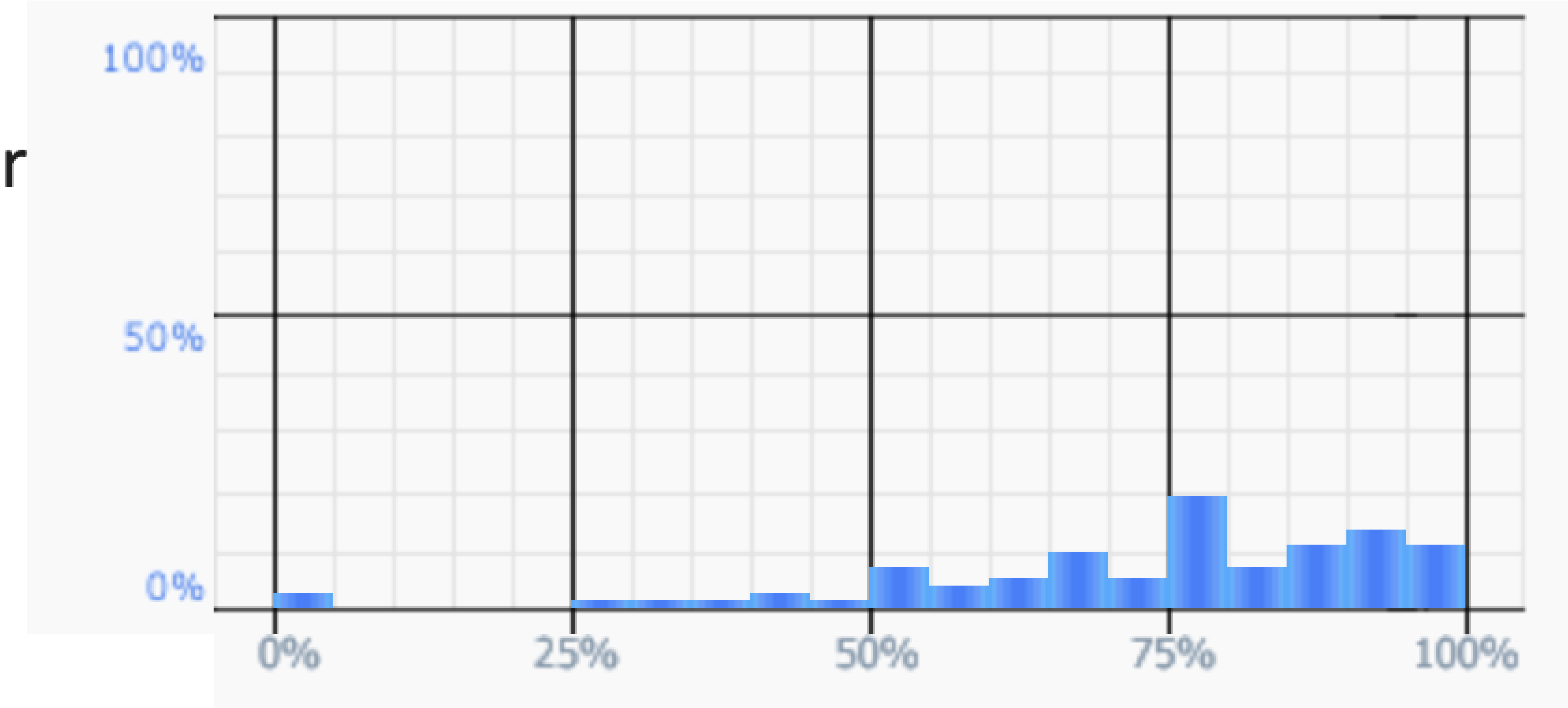
Mode: 77.5 %

Median: 77.5 %

Standard Deviation: 20.44 %

Grade Distribution

Number of Users (%)



Today's draft problem A

To be presented by today's A draftee.

Choose a number X uniformly at random from the set of numbers $\{1,2,3,4,5\}$. Now chose a number at random from the subset no larger than X (that is, from $\{1,\ldots,X\}$). Call this second number Y.

a) Find the joint PMF of X and Y
b) For each $i = 1,2,3,4,5$, find the conditional PMF of X given that $Y = i$.
c) Are X and Y independent?

Today's draft problem B

To be presented by today's B draftee.

Suppose that a university sells 225 permits for a student parking lot with 190 spots, and the students are independent and identically distributed, wanting to park in the lot 80% of the time. Use the central limit theorem to compute the probability that at least one student won't be able to park in the lot. (Hint: use continuity correction).

Question of the day – almost the same as Monday!

We're going to answer the following question for several examples today:

If X and Y are both continuous random variables and we understand their *joint* distribution (more specifically: their joint PDF), then what do their *conditional CDFs and PDFs* look like?

Definitions (unlike the discrete case these really are "new"!)

Conditional PDF (of X conditioned on Y)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)dx}$$

(assuming $f_Y(y) > 0$)

Conditional CDF (of X conditioned on Y)

Can use the conditional PDF to define the conditional CDF:

$$F_{X|Y}(a|b) = P\{X \leq a|Y = b\} = \int_{-\infty}^{a} f_{X|Y}(x|b)dx$$

More generally:

$$P\{X \in A|Y = b\} = \int_{A} f_{X|Y}(x|b)dx$$

Aside: isn't it illegal to condition on an event with probability 0???

Conditional CDF (of X conditioned on Y)

!!!

$$F_{X|Y}(a|b) = P\{X \leq a|Y = b\} = \int_{-\infty}^{a} f_{X|Y}(x|b)dx$$

What's really going on???

The integral on the right makes sense. What's going on is that the notation $P\{X \leq a|Y = b\}$ is a slight lie. The full truth is found in the following manipulations:

$$\int_{-\infty}^{a} f_{X|Y}(x|b)dx = \int_{-\infty}^{a} \frac{f(x,b)}{\int_{-\infty}^{+\infty} f(x,b)dx} dx = \frac{\int_{-\infty}^{a} f(x,b)dx}{\int_{-\infty}^{+\infty} f(x,b)dx} = \lim_{\epsilon\to 0^+} \frac{\int_{b-\epsilon}^{b+\epsilon}\int_{-\infty}^{a} f(x,b)dx\,dy}{\int_{b-\epsilon}^{b+\epsilon}\int_{-\infty}^{+\infty} f(x,b)dx\,dy}$$

Needs to be justified (can be via L'Hopital's rule + Leibnez integral rule…)

$$= \lim_{\epsilon\to 0^+} P(X \leq a\,|b - \epsilon \leq Y \leq b + \epsilon)$$

Okay for any $\epsilon > 0$

Aside: isn't it illegal to condition on an event with probability 0???

Conditional CDF (of X conditioned on Y)

$$F_{X|Y}(a|b) = P\{X \leq a|Y = b\} = \int_{-\infty}^{a} f_{X|Y}(x|b)dx$$

What's really going on???

In other words, if instead of writing $P\{X \leq a|Y = b\}$ we wrote $\lim_{\epsilon\to 0} P(X \leq a\,|b - \epsilon \leq Y \leq b + \epsilon)$, then everything would be fine!

However, writing $\lim_{\epsilon\to 0} P(X \leq a\,|b - \epsilon \leq Y \leq b + \epsilon)$ all the time would be annoying.

So, in the case that X and Y are jointly continuously distributed, we will just agree that $P\{X \leq a|Y = b\}$ is shorthand notation for $\lim_{\epsilon\to 0} P(X \leq a\,|b - \epsilon \leq Y \leq b + \epsilon)$.

This is sensible, because it's (arguably) the only "sensible" thing that $P\{X \leq a|Y = b\}$ could mean.

Google "Borel-Kolmogorov paradox" if you want to learn more.

Conditional PDF when X and Y are independent?

Conditional PDF (of X conditioned on Y)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{+\infty} f(a,y)da}$$

(assuming $f_Y(y) > 0$)

If X and Y are independent, this simplifies drastically. Do you see how?

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x)f_Y(y)}{f_Y(y)} = f_X(x)$$

Examples

Let's do Examples 5a and 5b on the chalkboard.

Friday's draft problem A

To be presented by Friday's A draftee.

The joint PDF of X and Y is

$$f(x,y) = \ f(x) = \begin{cases} c(x+y), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find c

b) Find $f_{X|Y}(x|y)$ for $0 < y < 1$

Friday’s draft problem B

To be presented by Friday’s B draftee.

Let X and Y be independent exponential random variables with the same rate λ. Find:

a) $f_{X|X+Y}(x|t)$

b) $F_{X|X+Y}(x|t)$