Lecture 14.1

More tricks w/ indicators and expectation

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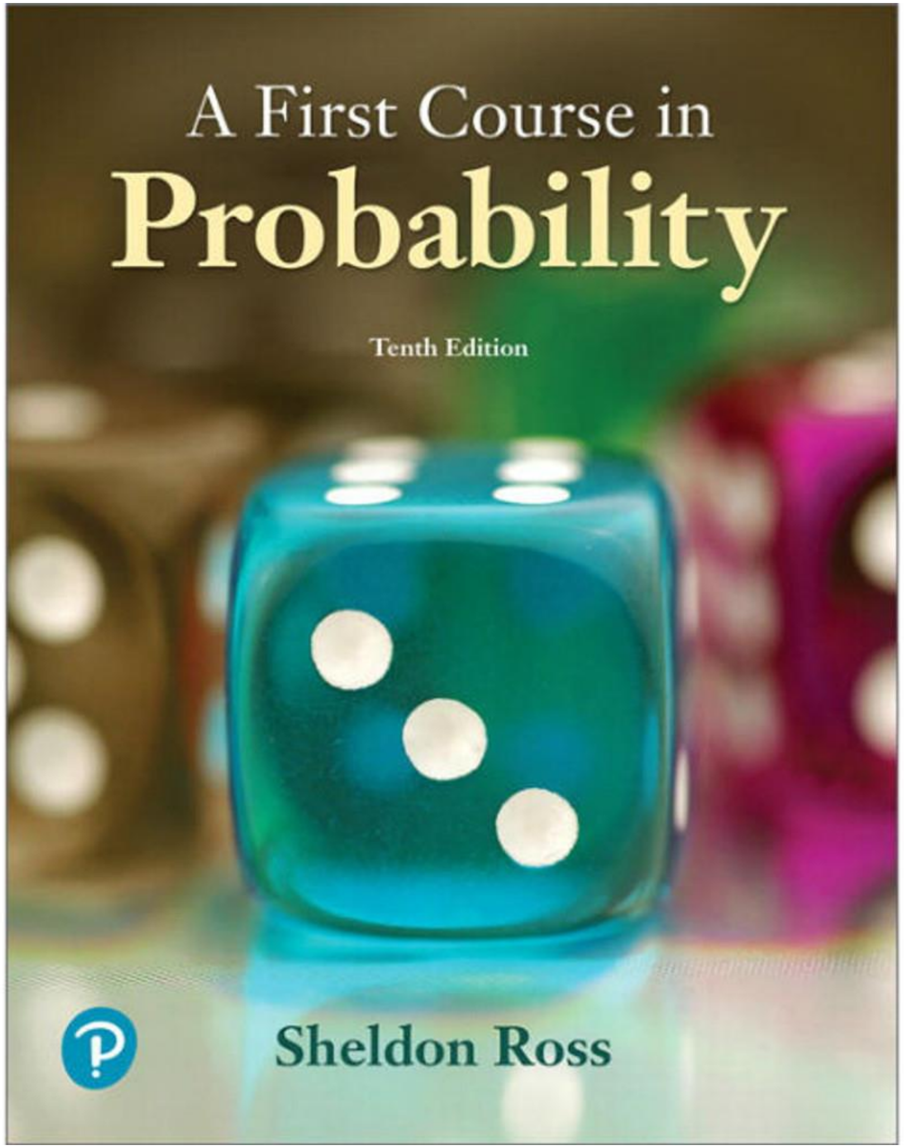


Today's reading: 7.3

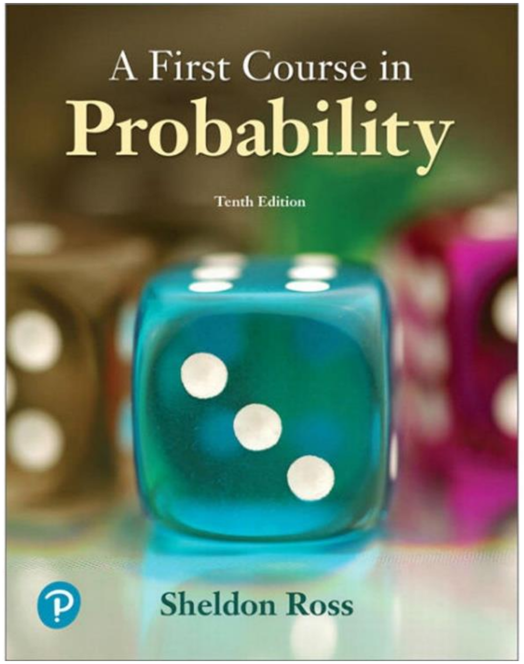
Next class: 7.4

HW11 is your LAST homework assignment!
Available now, due 4/25

Today's draft problem A

To be presented by today's A draftee.

Prove Proposition 2.1 (in Chapter 7) in the case that $g(x, y)$ is nonnegative and X and Y are jointly discrete.



Today's draft problem B

To be presented by today's B draftee.

Let X and Y be random variables with joint probability mass function defined by:

	$Y=-1$	$Y=0$	$Y=2$	$Y=6$
$X=-2$	$3/27$	$1/27$	$1/27$	$3/27$
$X=1$	$6/27$	0	$3/27$	$3/27$
$X=3$	0	0	$3/27$	$4/27$

a) Compute $E[XY]$
b) Determine if X and Y are independent.

Basic premise of the day:

If a random variable X counts how many events occur, then we can use (products of) indicator random variables to compute the expectation of $\binom{X}{k}$, the number of ways that k of the events occur.

We saw this last class for the case k=1:

Let $A_1, A_2, \ldots, A_n$ be events and for each $i = 1,2, \ldots, n$ define the indicator random variables

$$I_i = f(x) = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{else} \end{cases}$$

If $X = I_1 + \cdots + I_n$, then

$$E[X] = E[I_1] + \cdots + E[I_n] = P(A_1) + \cdots + P(A_n)$$

Basic premise of the day:

This generalizes as follows:

Let $A_1, A_2, \dots, A_n$ be events and for each $i = 1,2,\dots,n$ define the indicator random variables

$$I_i = f(x) = \begin{cases} 1, & \text{if } A_i \text{ occurs} \\ 0, & \text{else} \end{cases}$$

If $X = I_1 + \cdots + I_n$, then

$$\binom{X}{k} = \sum_{i_1 < i_2 < \cdots < i_k} I_{i_1} I_{i_2} \cdots I_{i_k}$$

and so

$$E[\binom{X}{k}] = \sum_{i_1 < i_2 < \cdots < i_k} E\left[I_{i_1} I_{i_2} \cdots I_{i_k}\right] = \sum_{i_1 < i_2 < \cdots < i_k} P(A_{i_1} A_{i_2} \cdots A_{i_k})$$

Basic premise of the day:

In particular, when k=2, we get a strategy for computing variances.

$$\binom{X}{2} = \frac{X(X-1)}{2} = \frac{X^2 - X}{2}$$

which implies

$$E[X^2] - E[X] = 2\sum_{i<j} E\left[I_i I_j\right] = 2\sum_{i<j} P(A_i A_j)$$

In particular, we get

$$Var(X) = \left(2\sum_{i<j} P\left(A_i A_j\right)\right) + E[X] - E[X]^2$$

Examples

3a: moments of binomial random variables via indicator random variables

3c: higher moments in the matching problem via indicators

3d: suppose there are N types of Pokémon, and for each $i = 1,2,\ldots,N$, in any encounter we have probability p_i of seeing a Pokémon of type i (where $\sum_{i=1}^{N} p_i$). If we have n encounters, then what are the expected value and variance of the number of types of Pokémon we see?

3f: suppose there are N types of Pokémon, and we have an equal likelihood of seeing anyone of them in a given encounter. Suppose we have encounters until we ”catch ‘em all.” Let’s find the expected number of types of Pokemon that we have caught EXACTLY ONCE. Let’s find the variance too.

Today’s draft problem A

To be presented by today’s A draftee.

Consider n independent flips of a coin having probability p of landing on heads. Say that a changeover occurs whenever an outcome differs from the one preceding it. (For instance, if $n = 5$ and the outcome is $HHTHT$, then there are 3 changeovers.) Find the expected number of changeovers.

Hint: Express the number of changeovers as the sum of $n - 1$ Bernoulli random variables.

Today's draft problem B

To be presented by today's B draftee.

Let $ABCD$ be the unit square where $A = (0,0), B = (1,0), C = (1,1), D = (0,1)$. Let $\alpha, \beta, \gamma, \delta$ be uniformly distributed on the intervals AB, BC, CD, DA. Let S be the area of the quadrilateral $\alpha\beta\gamma\delta$. Find $E[S]$.

Hint: $S = \frac{\det(\gamma-\alpha, \delta-\beta)}{2}$