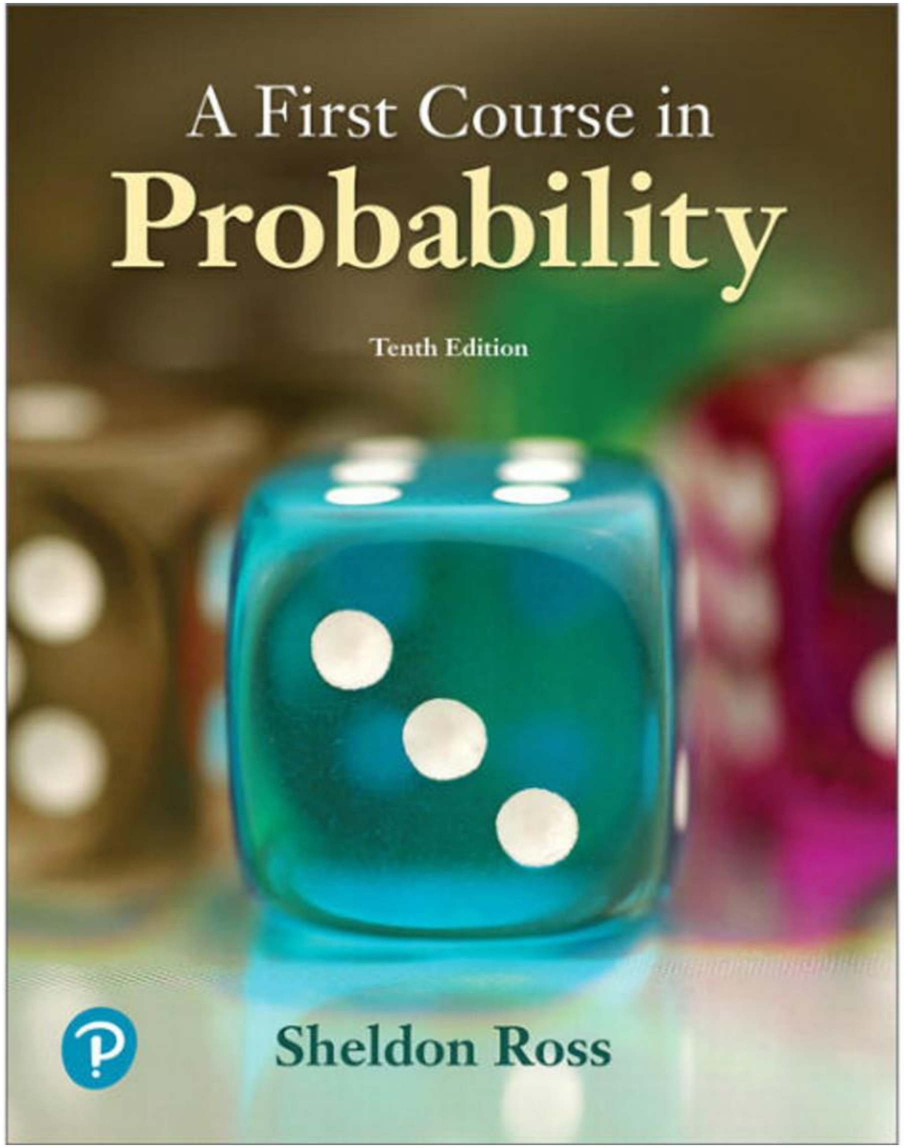
Lecture 14.3

Moment Generating Functions

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 7.7

Next class: 8.1-8.4

Last HW is due today!

Course evaluations are now open. ***Please do one!***

Next week: I've decided to rejigger the schedule.

- Monday we'll discuss CLT
- Wednesday I'll talk about quantum mechanics and have some time when you can "ask me anything"
- Friday we will not have class. Instead, I will record a video solving the practice final that you can watch on your own time.

Today's draft problem A

To be presented by today's A draftee.

9. A fair coin is tossed twice independently. Let X and Y be the indicator random variables of the events that "H" appear in the 1st and 2nd toss respectively.

(a) (5pts) Compute the covariance of $X+Y$ and $X-Y$ (simplify).

(b) (5pts) Are $X+Y$ and $X-Y$ independent? Justify your answer clearly.

Today's draft problem B

To be presented by today's B draftee.

Suppose X and Y are jointly continuously distributed with PDF

$$f(x,y) = \begin{cases} \dfrac{2e^{-2x}}{x}, & 0 \le x < \infty, 0 \le y \le x \\ 0, & else \end{cases}$$

Compute $Cov(X,Y)$.

Moment generating function

Definition

If X is a random variable, then its *moment generating function* $M_X : \mathbb{R} \to \mathbb{R}$ is the function

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots\right] \\ &= E[1] + tE[X] + \frac{t^2E[X^2]}{2!} + \frac{t^3E[X^3]}{3!} + \cdots \end{aligned}$$

Note: here t is just some $\mathbb{R}$-valued "formal" (i.e. "weird") variable, and I'm lying a little about $M_X(t)$ being a function on *all* of $\mathbb{R}$ (there are convergence issues…)

Moment generating function – WHY?!?

Seems kinda arbitrary??

Two key facts:

1. If we evaluate the n^{th} derivative of $M_X(t)$ at 0, then we get the n^{th} moment of X:

$$M_X^n(t) = E[X^n]$$

(Why?) This explains the name "moment generating function."

2. If X and Y are random variables with $M_X(t) = M_Y(t)$, then $X = Y$. In other words: the moment generating function uniquely determines the random variable! (We won't prove this, since we would want to iron out those convergence issues first...)

Somewhat deeper explanation: the moment generating function is the Laplace transform of the PDF...

Moment generating function – BUT REALLY WHY?!?

That still doesn't really answer the question!

Well, the moment generating function is yet another thing we can try to compute to better understand a given random variable. Since it completely determines the random variable and all of its moments, we should be satisfied if we're able to compute it!

Moreover, it comes with its own set of "formal properties" that can sometimes make it a more useful strategy for understanding a random variable than other strategies.

In particular, we will use moment generating functions to sketch the proof of the Central Limit Theorem on Monday.

Moment generating function – three useful observations

If X is discrete with PMF p, then:

$$M_X(t) = \sum_x e^{tx}\, p(x)$$

If X is continuous with PDF f, then:

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx}\, f(x)\, dx$$

(Why?)

If X and Y are independent, then the moment generating function for $X + Y$ is:

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

Moment generating functions – discrete examples

Table 7.1 Discrete Probability Distribution.

	Probability mass function, $p(x)$	Moment generating function, $M(t)$	Mean	Variance
Binomial with parameters n, p; $0 \le p \le 1$	$\binom{n}{x} p^x(1-p)^{n-x}$ $x = 0, 1, \ldots, n$	$(pe^t + 1 - p)^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda}\dfrac{\lambda^x}{x!}$ $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$ $x = 1, 2, \ldots$	$\dfrac{pe^t}{1-(1-p)e^t}$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$
Negative binomial with parameters r, p; $0 \le p \le 1$	$\binom{n-1}{r-1} p^r(1-p)^{n-r}$ $n = r, r+1, \ldots$	$\left[\dfrac{pe^t}{1-(1-p)e^t}\right]^r$	$\dfrac{r}{p}$	$\dfrac{r(1-p)}{p^2}$

Moment generating functions – continuous examples

Table 7.2 Continuous Probability Distribution.

	Probability density function, $f(x)$	Moment generating function, $M(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{e^{tb} - e^{ta}}{t(b-a)}$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\dfrac{\lambda}{\lambda - t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gamma with parameters $(s, \lambda), \lambda > 0$	$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{s-1}}{\Gamma(s)} & x \geq 0 \\ 0 & x < 0 \end{cases}$	$\left(\dfrac{\lambda}{\lambda - t}\right)^s$	$\dfrac{s}{\lambda}$	$\dfrac{s}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad -\infty < x < \infty$	$\exp\left\{\mu t + \dfrac{\sigma^2 t^2}{2}\right\}$	μ	σ^2