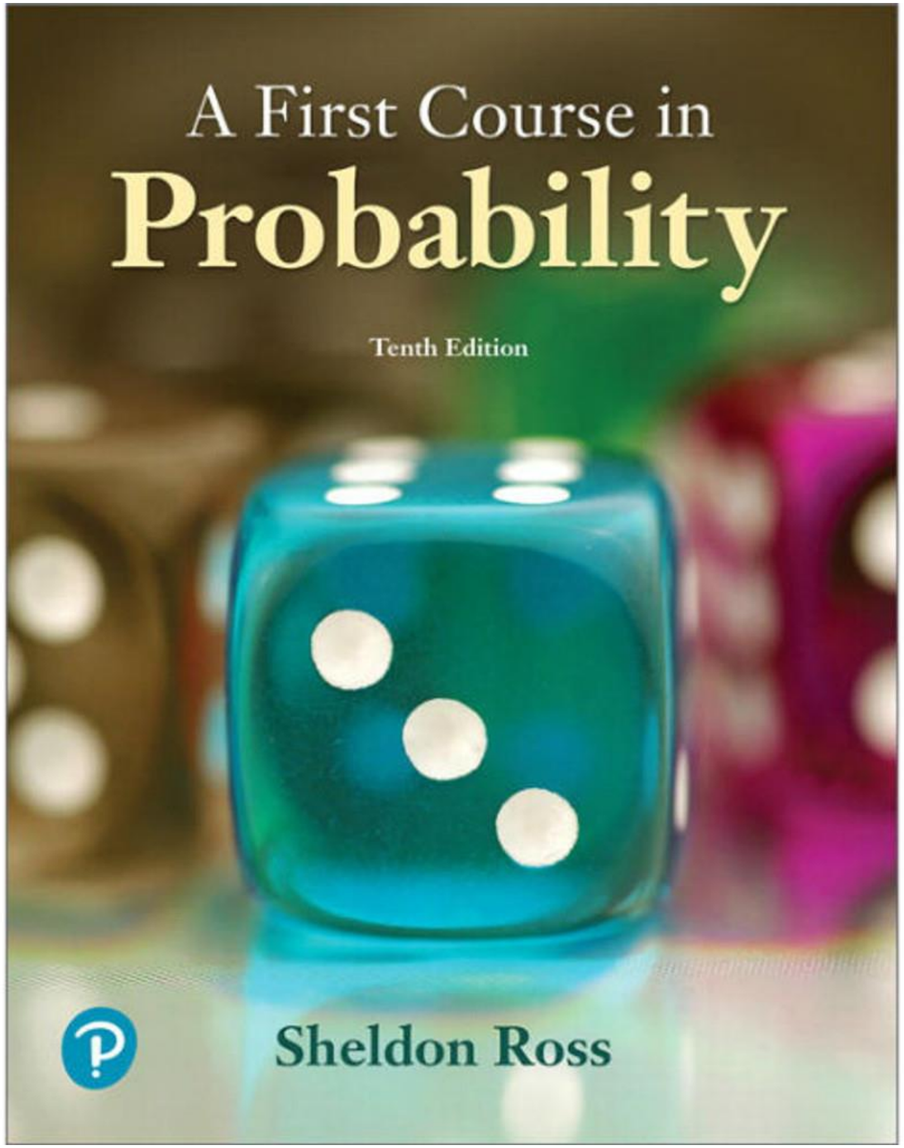
Lecture 2.3

Axioms of Probability

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 2.3+2.4

Reading for next class: 2.5+2.7 (can skip 2.6 if you want)

MA/STAT 416 Help Room is <u>NOW OPEN!</u>

Today's draft problem

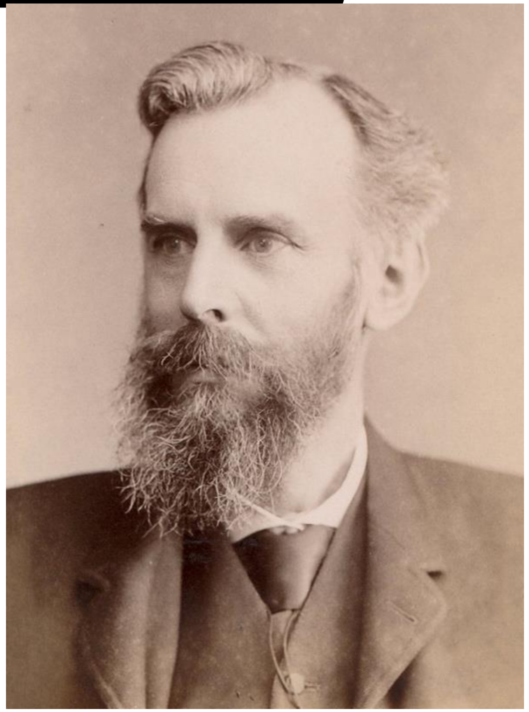
To be presented by today's draftee.

Using Venn diagrams:

a) Simplify the expression $(E \cup F)(E \cup F^c)$.

b) Show $(E \cup F)^c = E^c F^c$.

c) Show $(EF)^c = E^c \cup F^c$.



Recall: Sample space

Definitions:

The set of all outcomes of an experiment is called the *sample space*. Usually denoted S.

Given a sample space S, an *event* is any subset E of S. Denoted $E \subset S$.

Recall: mutually exclusive

Definition:

The (infinitely many) events $E_1, E_2, E_3, \ldots$ in S are *mutually exclusive* if $E_i E_j = \emptyset$ whenever $i \neq j$.

Axioms of Probability

Let S be a sample space. A ***probability measure*** on S is a function P that takes each event $E \subset S$ to a real number $P(E)$ (called the ***probability*** of event E) in a way that satisfies the following three axioms:

Axiom 1.

$$0 \leq P(E) \leq 1$$

Axiom 2.

$$P(S) = 1$$

Axiom 3.

For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Axioms of Probability: (Bayesian) interpretation

Axiom 1.

$$0 \leq P(E) \leq 1$$

Axiom 2.

$$P(S) = 1$$

Axiom 3.

For any sequence of mutually exclusive events $E_1, E_2, E_3, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Interpretation of Axiom 1. The probability that an event E occurs is given by a number between 0 and 1. This number (the probability) represents a "*quantification of belief*" about whether E will occur or not.

Interpretation of Axiom 2. The probability that *something* will happen is 1. In other words—we can be *certain* that performing the experiment will result in an outcome in the sample space. (This of course assumes we have correctly mathematically modeled our experiment's sample space.)

Intuition for Axiom 3. Probabilities of mutually exclusive events add.

THAT'S IT!

Of course, these three simple little axioms can be used to do a LOT.

Axioms of Probability: some simple applications

Proposition 4.1. $P(E^c) = 1 - P(E)$

Proposition 4.2. If $E \subset F \subset S$, then $P(E) \leq P(F)$.

Proposition 4.3. If $E \subset S$ and $F \subset S$, then

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

Note: in math, a "Proposition" is like a "Theorem" (just not as interesting or powerful).

Let's use the Axioms of Probability to prove each of these propositions on the chalkboard.

"Inclusion-Exclusion Identity": generalizing Prop 4.3

Proposition 4.4.

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cdots E_{i_r})$$

Let's do the example where $n = 3$, i.e. for $P(E_1 \cup E_2 \cup E_3)$.

Please read 2.4 for more info.

Monday's draft problem

To be presented by the draftee at the beginning of class on Monday.

1. Unpack the inclusion-exclusion identity in the case $n = 4$. That is, determine the explicit formula for $P(E_1 \cup E_2 \cup E_3 \cup E_4)$.

2. Prove that for any probability measure P on any sample space S, we have
$$P(\emptyset) = 0.$$