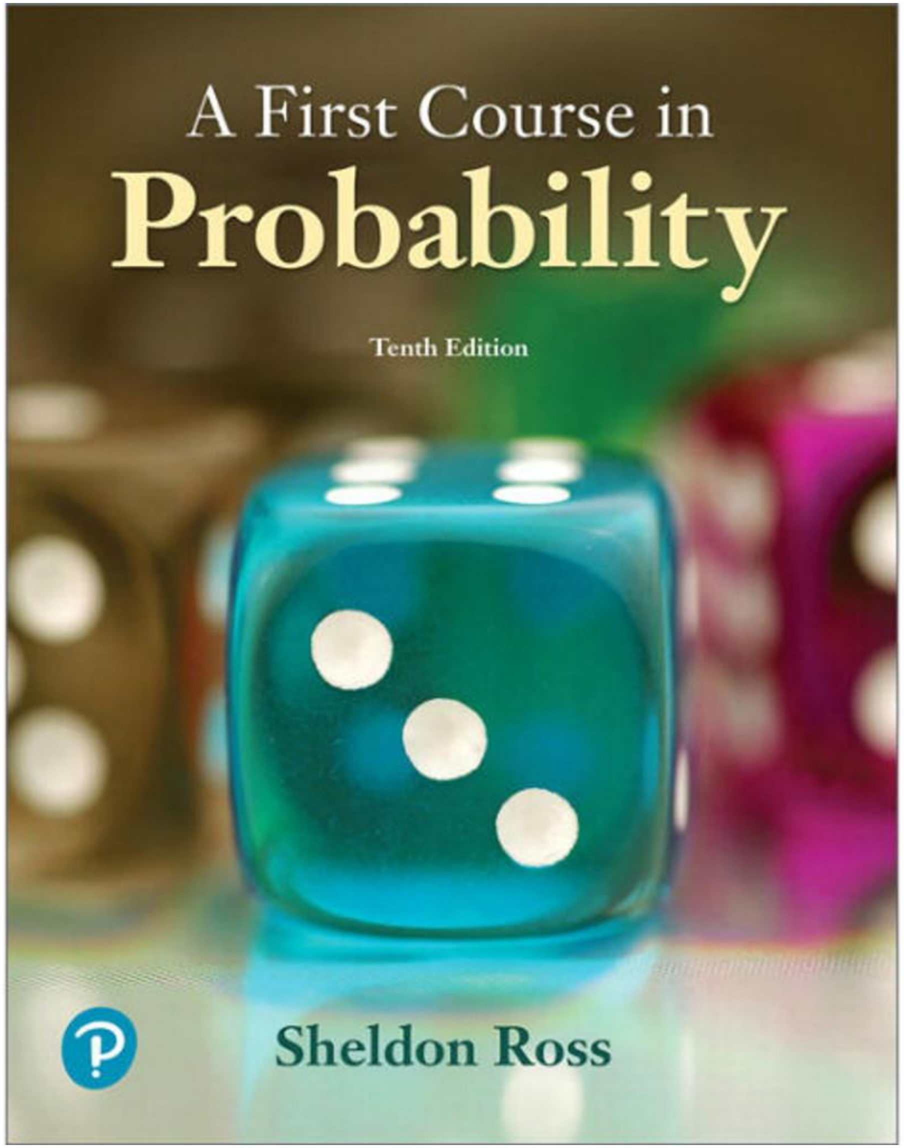
Lecture 4.2

Bayes's Theorem

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 3.3

Reading for next class: 3.4

MA/STAT 416 Help Room is <u>NOW OPEN!</u>

HW3 is due Friday.

Today's draft problem

To be presented by today's draftee.

House is certain that his patient either has an autoimmune disease or an infection, but not both. Based on his patient's symptoms, House is 90% sure that the patient has an autoimmune disease and thus will treat the disease with steroids. If the patient really does have an autoimmune disease, then the probability that they die after receiving the treatment is 15%. If the patient does not have an autoimmune disease, then the probability that they die after receiving the treatment is 100% (because the steroids will allow the infection to run rampant). Dr. Cuddy will not let House perform the treatment unless the probability that the patient will survive it is at least 85%. Does she let House perform the treatment?

meirl?

E.g., when I say stuff like "Bayes theorem is such an important idea that it can be used as the basis for an entire philosophy of how to interpret the laws of probability."

https://www.smbc-comics.com/comic/precise

Easy version of Bayes's theorem - statement

(Not made explicit in book):

Bayes's Theorem
If P is any probability measure on a sample space S and E and H are two events in S with $\mathrm{P}(E) > 0$, then

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Recall that we need $P(E) > 0$ to be able to divide by it!

Easy version of Bayes's theorem - intepretation

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)} = \frac{P(E|H)}{P(E)} P(H)$$

Updated probability
of the hypothesis
given new evidence

Original probability
of our hypothesis

In light of the new evidence, we
should update our original
probability of the hypothesis by
multiplying by this "Bayes factor"

Easy version of Bayes's theorem - proof

On one hand:

$$P(EH) = P(E|H)P(H)$$

On the other hand:

$$P(EH) = P(HE) = P(H|E)P(E)$$

Thus:

$$P(E|H)P(H) = P(H|E)P(E)$$

and so

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

QED

A definition we need before “full Bayes’s theorem”

Let S be a sample space with events $H_1, H_2, \dots, H_n$.
We say these events are <u>*exhaustive*</u> if

$$S = \bigcup_{i=1}^{n} H_i$$

(Note: a set of events in S that is *both* mutually exclusive *and* exhaustive is the same thing as a partition of S. (If you don’t know what a partition is, don’t worry, this is just for fun.))

A fact we need before "full Bayes's theorem"

Law of Total Probability
If P is any probability measure on a sample space S, $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive events in S, and E is another event, then

$$P(E) = \sum_{i=1}^{n} P(EH_i) = \sum_{i=1}^{n} P(E|H_i)P(H_i)$$

Proof on chalkboard

Full version of Bayes's theorem - statement

From the book:

Bayes's Theorem
If P is any probability measure on a sample space S, $H_1, H_2, \dots, H_n$ are mutually exclusive and exhaustive events in S, and E is another event, then

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{\sum_{j=1}^{n} P\left(E\middle|H_j\right)P\left(H_j\right)}$$

Proof on chalkboard

Friday's draft problem

To be presented by Friday's draftee at the beginning of class.

An expensive painting is stolen. ICE knows that the thieves have hidden it on one of four container ships that is coming into the port of Hamburg, but they don't know which one, so they assume all four possibilities are equally likely. If the painting is in fact on the i^{th} ship, then the probability that ICE will find it is $1 - \beta_i$ (i=1,2,3,4) for some $0 < \beta_i < 1$ (the "overlook probability"). For each i=1,2,3,4, determine the conditional probability that the painting is on ship i given that ICE searches ship 1 and finds nothing.