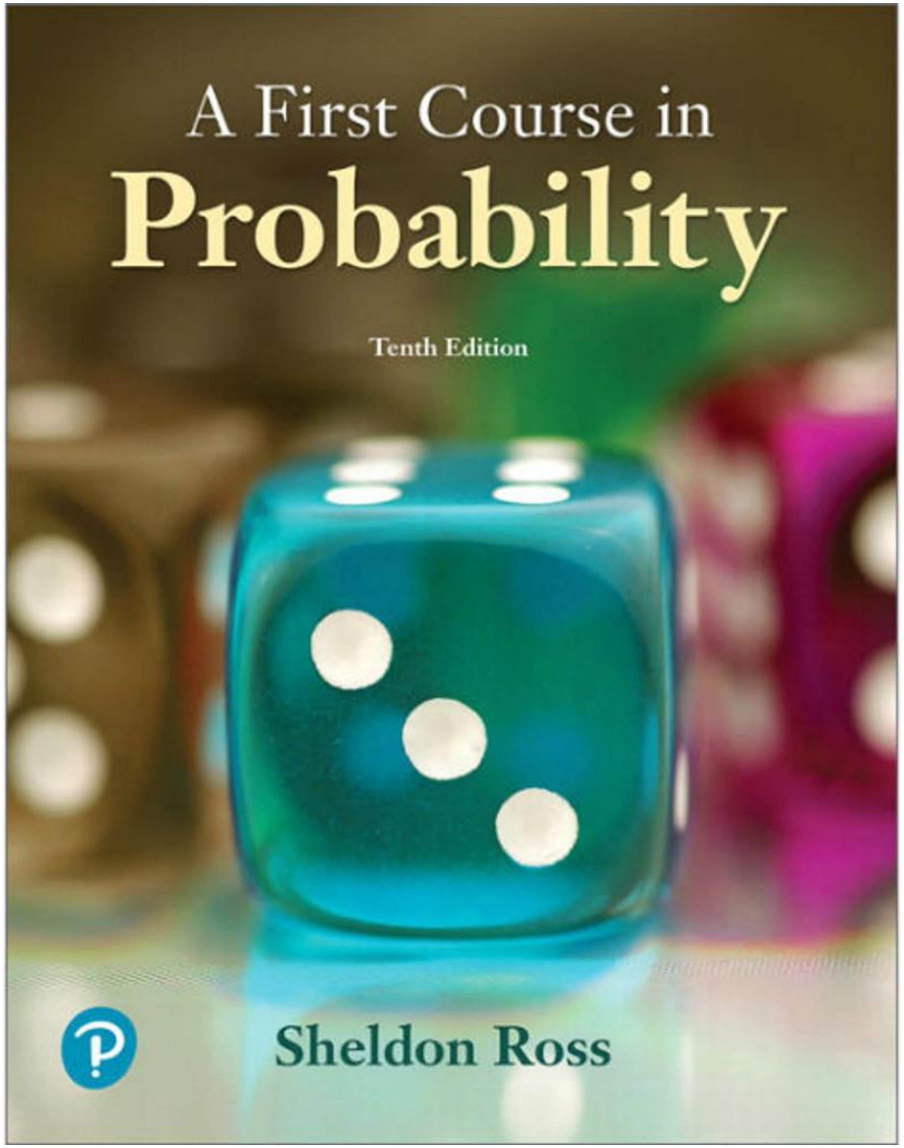
Lecture 6.2

(Real-valued) random variables

Department of Mathematics



Today's reading: 4.1+4.2

Next class: 4.3

No HW this week. HW5 will be made available soon.

I will do my best to return graded MT1 in class on Monday.

(Real-valued) random variables

Definition (more-or-less from the book):

Let S be a sample space. A *(real-valued) random variable on* S is a function

$$X: S \to \mathbb{R}$$

Comments:

- *REMEMBER ALL OF THIS ASAP.*
- Notice that the definition has NOTHING to do with any choice of probability measure on S. (Recall: a given sample space S typically has *many* different probability measures on it.) More on this momentarily.
- If T is any set, we could talk about an "T-valued random variable on S". This would simply be a function $\mathrm{X}: S \to T$.

Random variables – why do we care?

Let S be a sample space. A *(real valued) random variable on S* is a function

$$X\colon S \to \mathbb{R}$$

Practically:

We often don't care about the specifics of events in S, but instead have some way of associating a real number to every outcome in S. E.g., if we roll two dice but only care about their sum, then we really only care about a single real number between 2 and 12 (rather than a pair of numbers, each between 1 and 6).

Formally/mathematically:

If we have a random variable $X\colon S \to \mathbb{R}$ ***and*** we have a probability measure P on S, then we can use X to "pushforward" P and define a probability measure on $\mathbb{R}$. (By a small "abuse of notation," we often denote this measure on $\mathbb{R}$ by P, even though we should really call it something like X_*P.)

[general construction on chalkboard (*I'm showing you this for your benefit, but you do not need to remember it!*)]

Intuitively: real-valued random variables allow us to convert probability measures on "weird" sample spaces S into probability measures on something we know and love, namely, the real line $\mathbb{R}$.

Cumulative distribution functions

Let S be a sample space. A *(real valued) random variable on* S is a function

$$X\colon S \to \mathbb{R}$$

Definition:

Fix a sample space S with a probability measure P and let $X: S \to \mathbb{R}$ be a random variable. The *cumulative distribution function* of X is the function $F: \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = P(\{X \leq x\})$$

Comments:

- The definition of the cumulative distribution function of X depends on X, S and P, even though the notation does not make this clear. *Sorry, but get used to it!*
- We often refer to the cumulative distribution function as the "CDF," or drop the word "cumulative" and just say "distribution function."
- If I had my druthers, I would denote the CDF by $\mathrm{CDF}_{S,P,X}$
- The CDF is monotone increasing. (Why?)
- We will see later why this is useful.

THE VIEW FROM 10,000 FEET: A FUNDAMENTAL FACT ABOUT PROBABILITY MEASURES ON $\mathbb{R}$

Every probability measure on $\mathbb{R}$ is a "mixture" of a "discrete probability measure" and a "continuous probability measure."

Comments:

- This is called the "Lebesgue decomposition theorem" (I haven't stated it precisely, but I think it is helpful to see, since it gives some understanding of why we care so much about "discrete" vs "continuous" random variables.
- We won't prove this theorem (take MA 538 or MA 544 if you want to).
- We will define "discrete probability measures" momentarily (and, more importantly, "discrete random variables").
- Chapter 4 only considers discrete random variables. (Thus, we will only consider discrete random variables for the next 3 weeks.)
- In Chapter 5, we will study "continuous probability measures" on $\mathbb{R}$ and, relatedly, "continuous random variables."

Discrete probability measures on $\mathbb{R}$

Definition (NOT in the book, but hopefully helpful):

A probability measure P on $\mathbb{R}$ is <u>*discrete*</u> if there exists a finite or at-most *countably infinite* set of real numbers $x_1, x_2, x_3, \ldots \in \mathbb{R}$ with the two following properties:

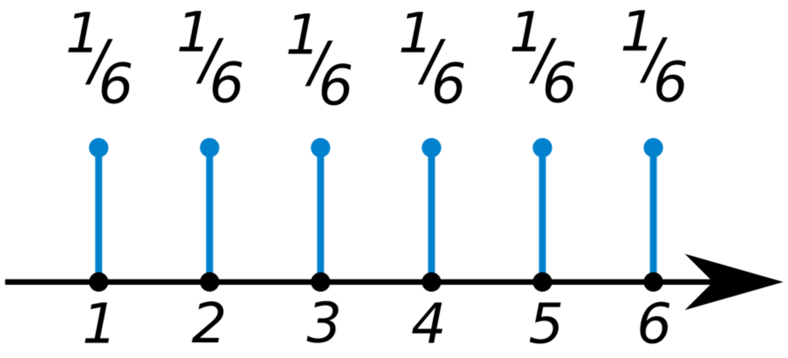
- $P(x_i) > 0$
- $\sum_{i=1}^{\infty} P(x_i) = 1$

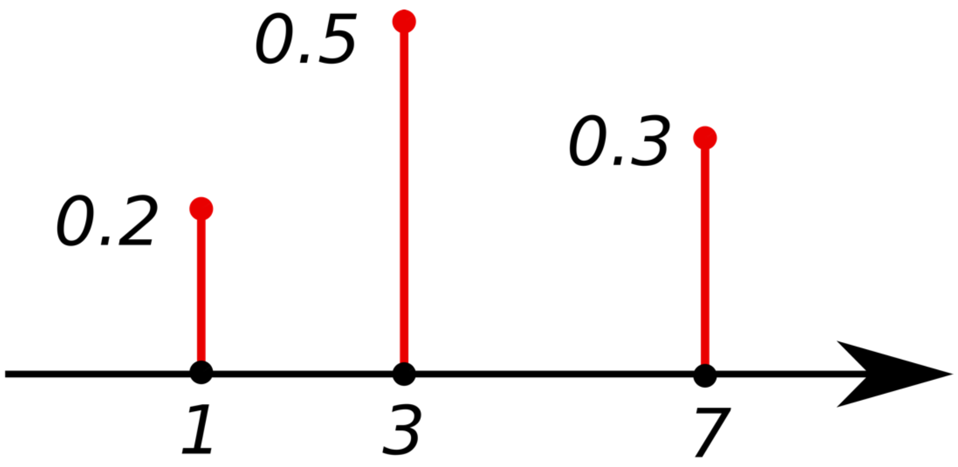
Intuition/equivalently: a probability measure P on $\mathbb{R}$ is discrete if the probability of an event $\mathrm{E} \subset \mathbb{R}$ is the sum of the probabilities of the individual outcomes in E. That is: $\mathrm{P}(E) = \sum_{e \in E} P(\{e\})$

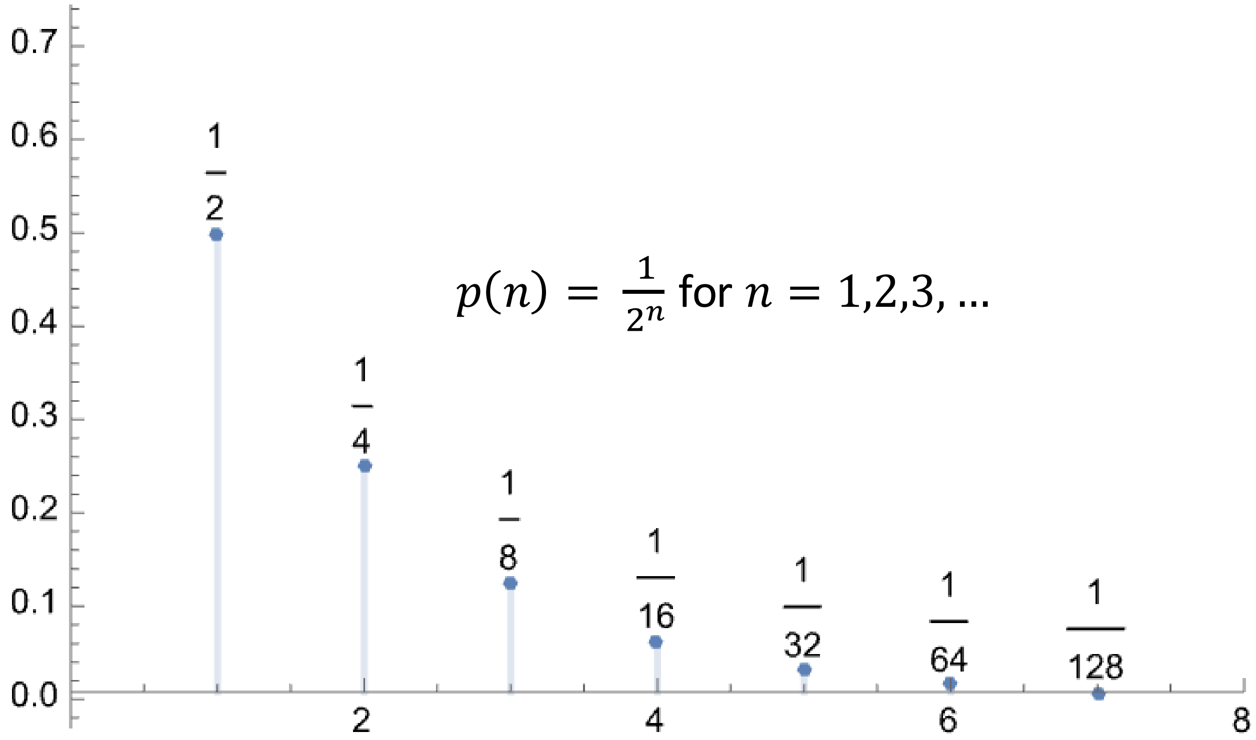
If P is discrete, then its <u>*probability mass function (or PMF)*</u> is the function $p: \mathbb{R} \to \mathbb{R}$ defined as $p(x) = P(\{x\})$. Being discrete guarantees that P is entirely determined by its probability mass function.

Discrete probability measures on $\mathbb{R}$: examples

We specify these examples by their PMFs







Looking ahead: an example of a continuous probability measure on $\mathbb{R}$

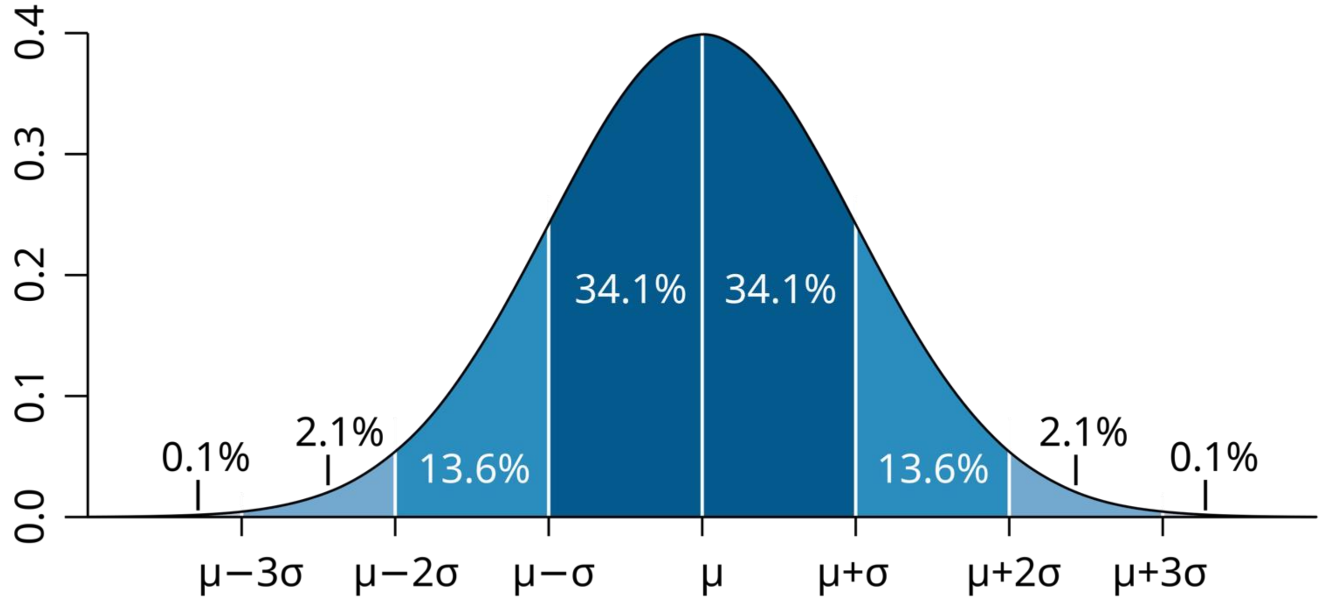
"Absolutely continuous" probability measures on $\mathbb{R}$ are specified by a "probability density function" (PDF).

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If the PDF is $f\colon \mathbb{R} \to \mathbb{R}$, then the probability of an event $E \subset \mathbb{R}$ is

$$P(E) = \int_{x \in E} f(x)\, dx$$

In particular, for a continuous probability measure, the probability of a single outcome is always 0. (Contrast with discrete measures!)



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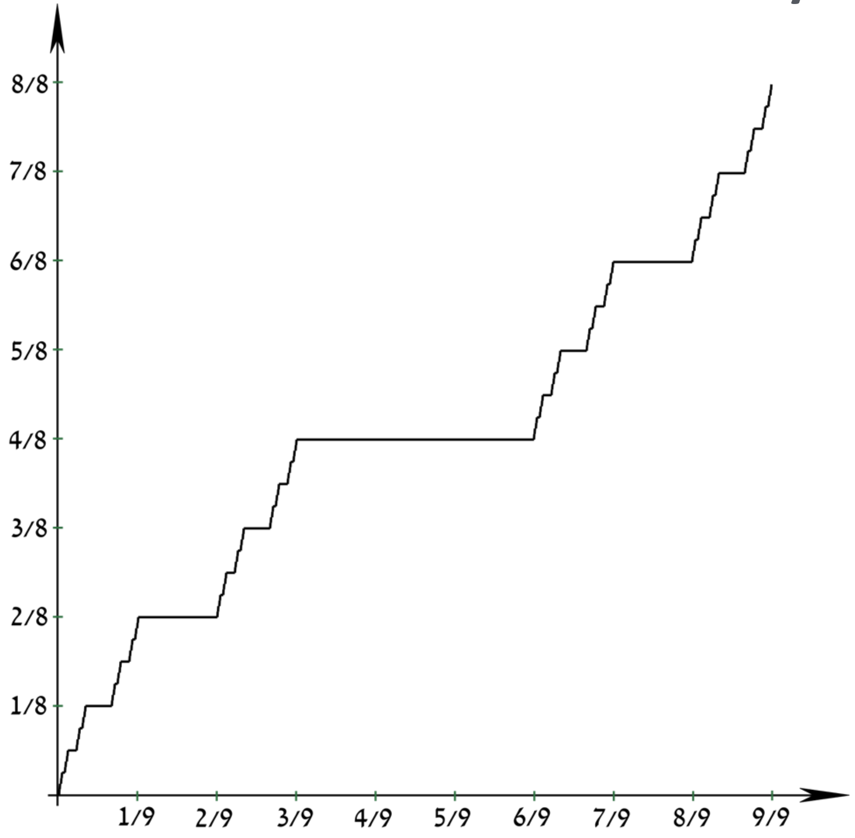
Cantor distribution: a continuous probability measure on $\mathbb{R}$ with a CDF but no PDF

This example is "pathological" and not of the type we will study later. (It is "continuous" but not "absolutely continuous." This is just for fun!)



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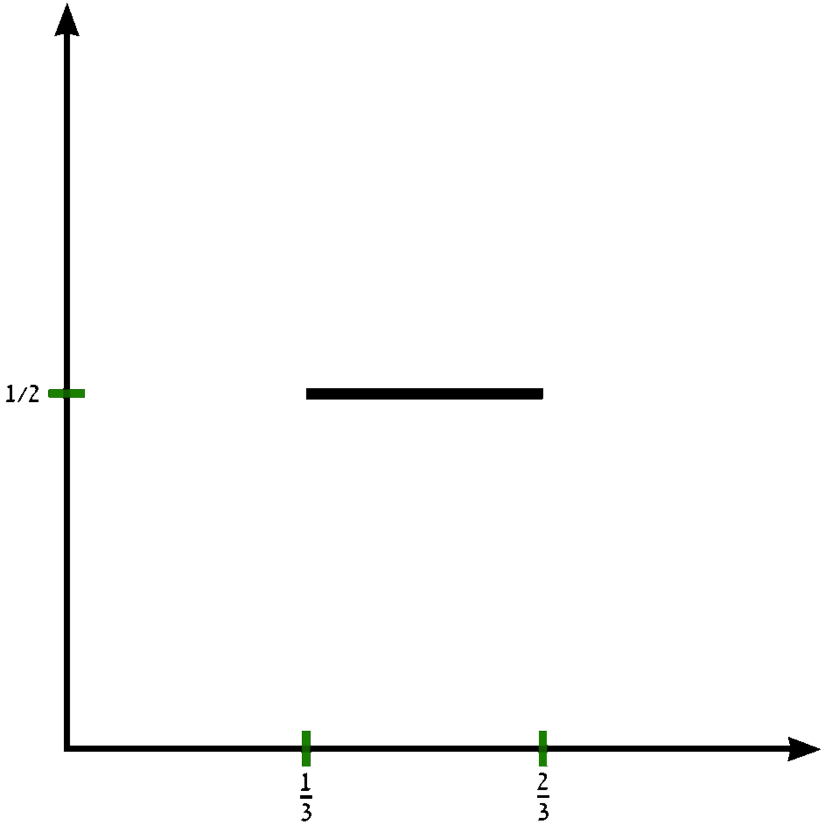

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Discrete random variables

Definition:

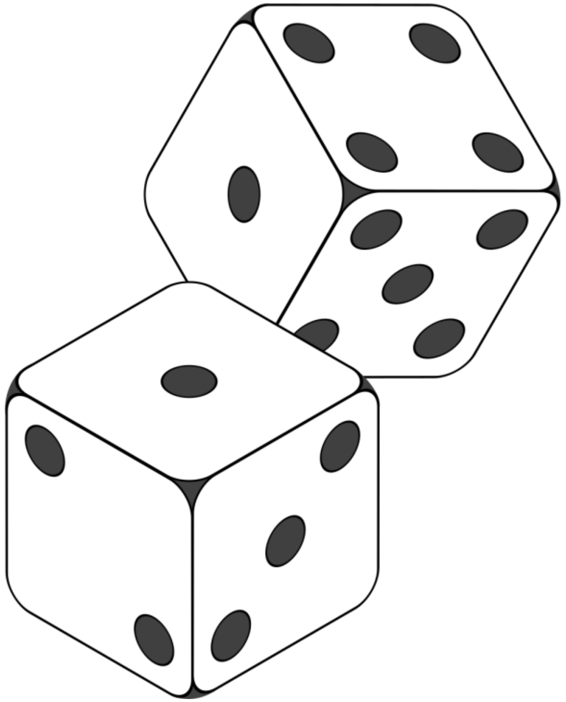
A (real-valued) random variable $X: S \to \mathbb{R}$ is *discrete* if it takes on at most countably infinitely many different values in $\mathbb{R}$.

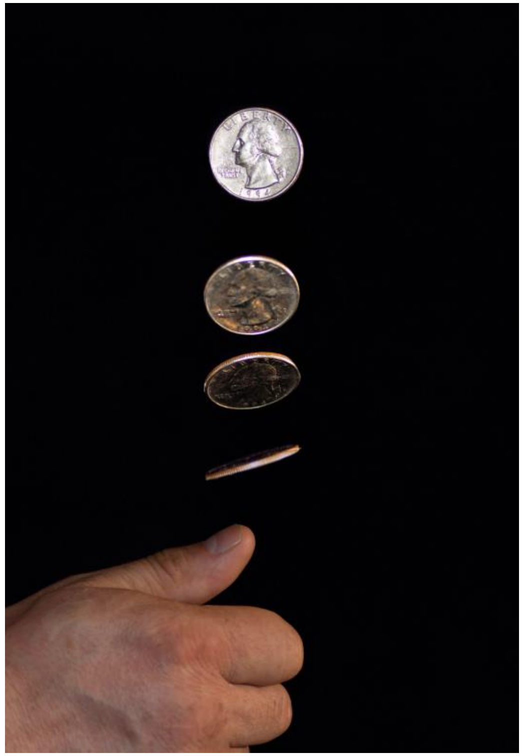
Equivalently, $X: S \to \mathbb{R}$ is discrete if for every probability measure P on S, the induced probability measure on $\mathbb{R}$ is discrete.

We will see many, many examples in the next 3 weeks.

Friday's draft problem

To be presented by Friday's draftee.





Two fair dice are rolled and one fair coin is tossed. Let X be the product of the two dice outcomes together with +1 if the coin was heads or -1 if the coin was tails. Compute $P\{X = i\}$for each $i = \pm 1, \pm 2, \ldots, \pm 36$.