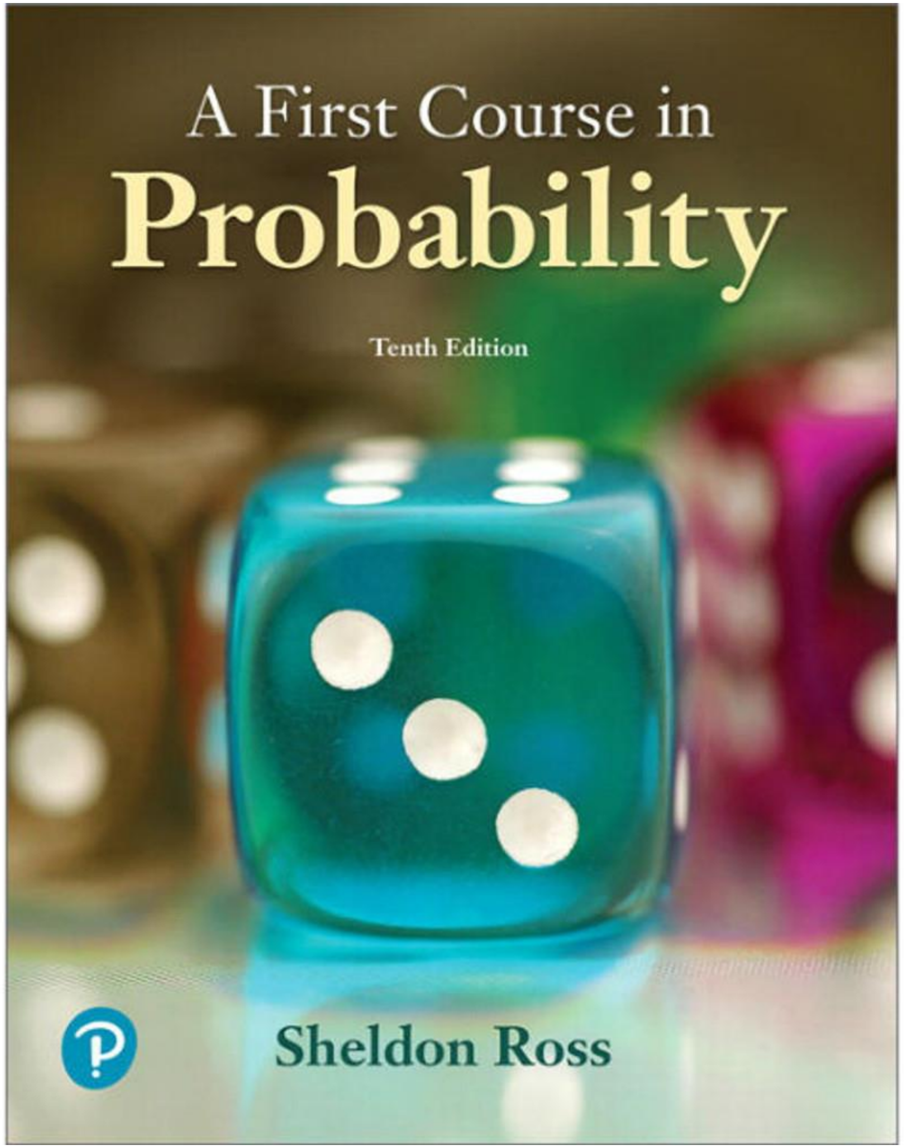
Lecture 8.1

Bernoulli and Binomial Random Variables

PURDUE UNIVERSITY | Department of Mathematics



Today's reading: 4.6

Next class: 4.7

HW6 now available

Today's draft problem

To be presented by today's draftee.

Fix a positive integer k. Purdue and IU are going to play a series of sportsball games until one of them wins k of them. The probability that Purdue wins any one game is $p \geq 0.5$, and the results of different games are independent of one another.

(a) Let X be the random variable that counts the number of total games played. If $k = 2$, find the variance of X.

(b) What is the value of p that maximizes this variance? (Continue to assume $k = 2.$)

Recall - Bernoulli random variables

Definition (from the book, more-or-less):

A *Bernoulli random variable* is a random variable $X: S \to \mathbb{R}$ that only takes the values 0 and 1. In other words, a Bernoulli random variable is a random variable with a pmf that looks like

$$p(0) = P\{X = 0\} = 1 - p$$
$$p(1) = P\{X = 1\} = p$$

for some p.

Intuition

A Bernoulli random variable is like the flip of a single, biased coin, where the probability of Heads (“Success” or “1”) is p.

Binomial random variables

Definition (from the book, more-or-less):

A ***binomial random variable*** with parameters (n, p) is the random variable we get by performing n independent trials of a Bernoulli random variable whose success probability is p.

In other words:

A binomial random variable w/ parameters (n, p) counts the number of heads we see if we flip a biased coin exactly n times.

Note:

If $n = 1$, then a binomial random variable w/ parameters (n, p) is just a Bernoulli random variable.

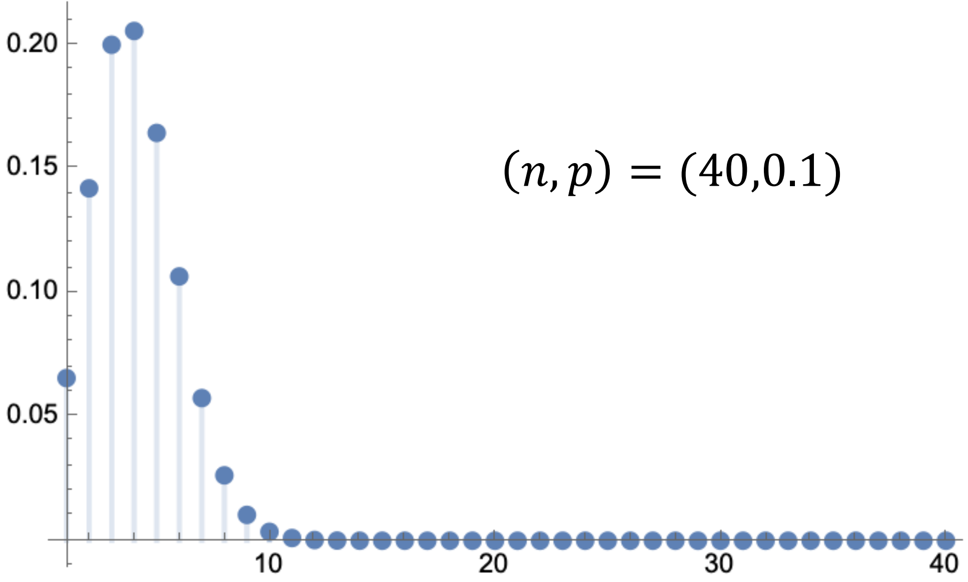
PMF of a binomial random variable

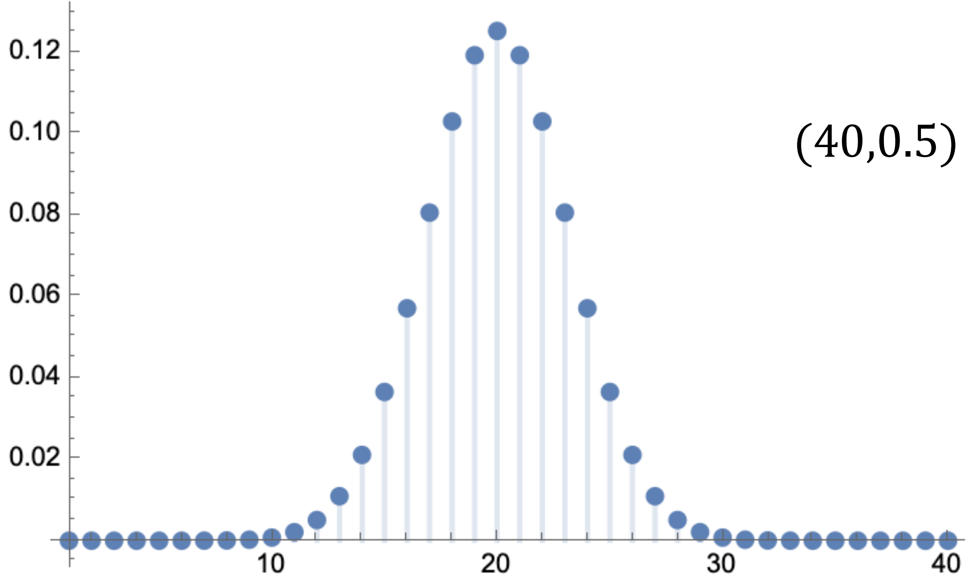
Let X be a binomial random variable with parameters (n, p). Then the PMF of X is the function p with

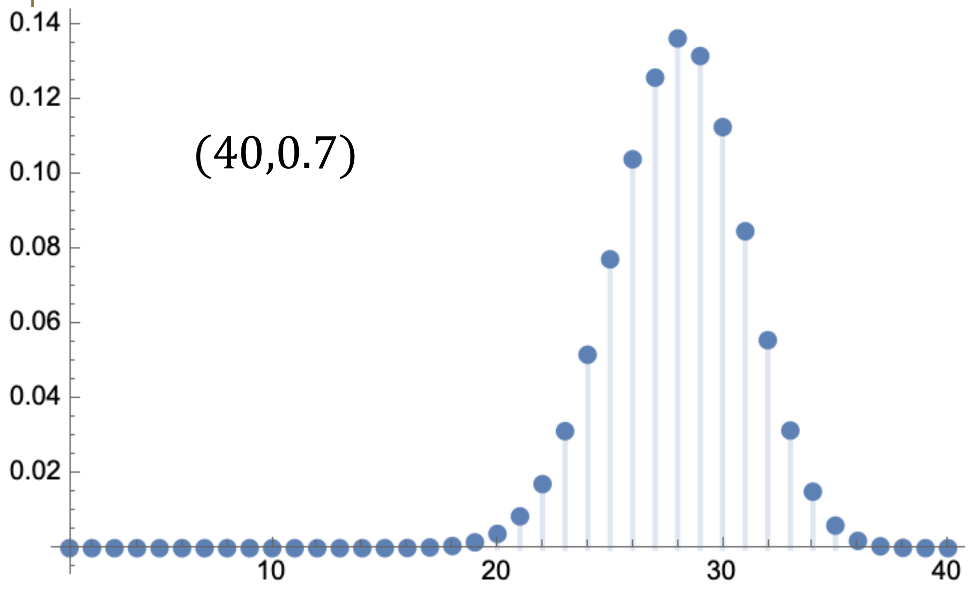
$$p(i) = \binom{n}{i} p^i (1-p)^{n-i} \qquad\qquad i = 0, 1, \ldots, n$$

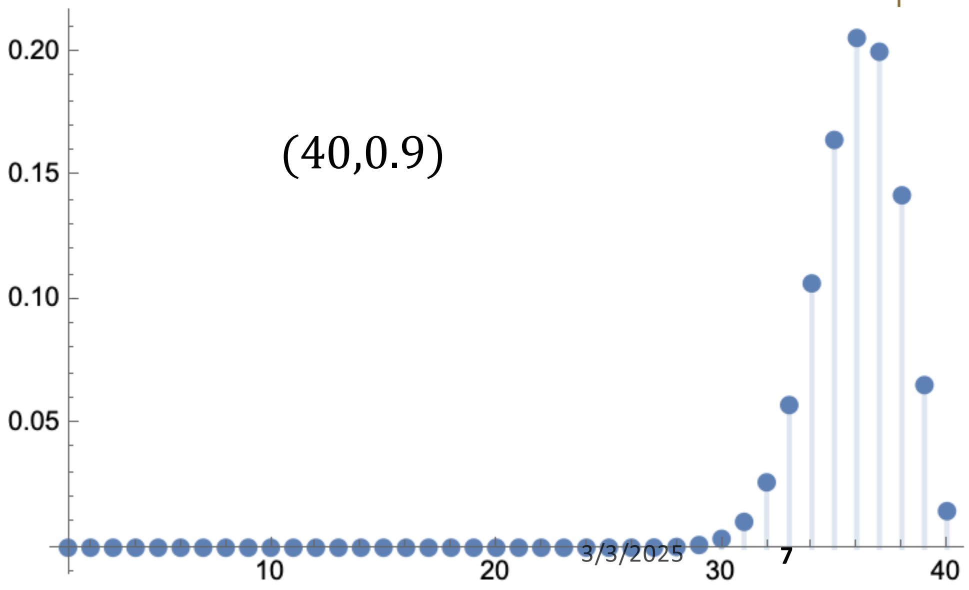
Why is this a valid PMF?

Let's plot the PMF for $(4, 0.5)$


$(n, p) = (40, 0.1)$
0.20
0.15
0.10
0.05
10
20
30
40


$(40, 0.5)$
0.12
0.10
0.08
0.06
0.04
0.02
10
20
30
40


$(40, 0.7)$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
10
20
30
40


$(40, 0.9)$
0.20
0.15
0.10
0.05
10
20
30
40

Expectation and variance of a binomial random variable

If X is a binomial random variable with parameters (n, p), then

$$E[X] = np$$

$$Var(X) = np(1-p)$$

Let's prove it on chalkboard.

The "peak" of a binomial random variable's PMF

Proposition 6.1

If X is a binomial random variable with parameters (n, p), where $0 < p < 1$, then as k goes from 0 to n, $\mathrm{p(k)} = P\{X = \mathrm{k}\}$ increases monotonically at first, then decreases monotonically, reaching its largest value when k is the largest integer less than or equal to $(n+1)p$.

Intuition

The PMF of a binomial random variable is maximized at the first integer situated just above its mean.

What about $\boldsymbol{p = 0}$ or $\boldsymbol{p = 1}$?

Wednesday’s draft problem

To be presented by Wednesday’s draftee.

This is a variation of Example 6c in Chapter 4 of Ross.

Three, fair, **four-sided** dice (number 1,2,3,4) are to be tossed, and a player bets on one of the numbers 1 through 4. If the number bet by the player appears i times, $i = 1,2,3$, then the player wins i dollars; if the number bet by the player does not appear, then they lose 1 dollar. Should the player play? (That is, if they do play, should they expect to make money or lose money?)