Lecture 8.3

Poisson Processes; Linearity of expectation

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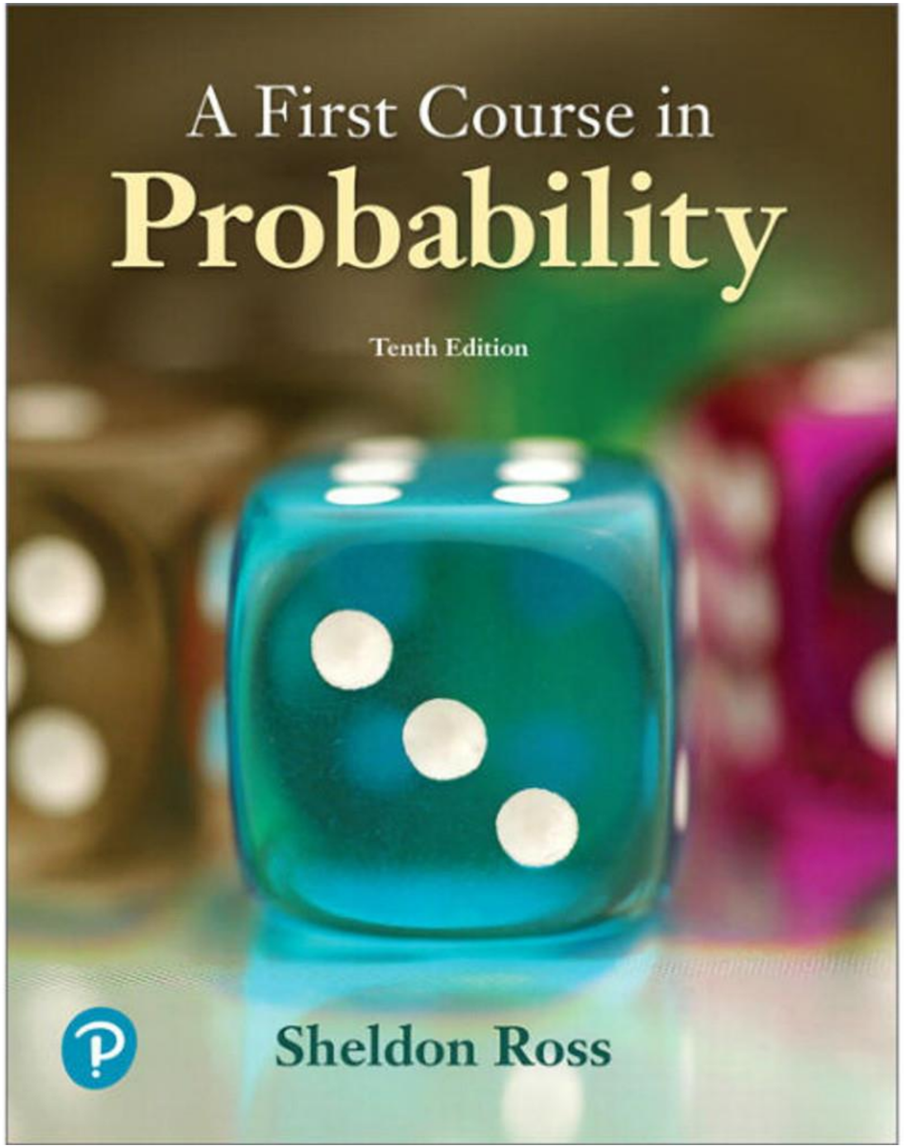

Today's reading: 4.7+4.9 (you can skip 4.8 for now, although we might need to come back to some of it later. You should look at 4.10, but don't worry too much about it for now.)

Next class: 5.1, 5.2, 5.3

HW6 due today

Today's draft problem

To be presented by today's draftee.

Suppose that the number of accidents that occur on a highway each day is a Poisson random variable X with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs. (In other words, compute the conditional probability $P(\{X \geq 3\}|\{X \geq 1\})$.

Recall = Poisson random variables

Definition (from the book):

A *Poisson random variable* with parameter λ is a random variable X that takes the values $0, 1, 2, \ldots$ and has PMF given by

$$p(i) = P\{X = i\} = e^{-\lambda}\frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

Recall - Poisson random variables - WHY?!

Two explanations:

1. If X is a binomial random variable with parameters (n, p) where n is "large" and p is "small enough" so that $\mathrm{E}[X] = np$ is "not large," then a Poisson random variable with parameter $\lambda = np$ is a "good approximation" to X.

2. Poisson PMFs show up in situations where we want to count the number of times a relatively rare discrete event occurs in a large, continuous interval of time.

Today

Little o notation

Definition (more-or-less from the book):

A function $f(h)$ is a said to be "little o of h," sometimes written $\mathrm{f} = o(h)$, if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

For example: $\mathrm{f}(h) = h^2$ is $\mathrm{o}(h)$ because

$$\lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0$$

Poisson process - definition

Definition (more-or-less from the book):

A *Poisson process* with rate λ is a random process where discrete "hits" can happen repeatedly at specific moments in time (which is taken to be continuous) subject to the following requirements:

1. The probability that exactly 1 hit occurs in an interval of time of length h is $\lambda h + o(h)$.
2. The probability that 2 or more hits occur in an interval of time of length h is $\mathrm{o}(h)$.
3. For any integers $n, j_1, j_2, \dots, j_n$ and any set of n non-overlapping intervals of time, if E_i denotes the event where exactly j_i hits occur in the i^{th} interval, then the events $E_1, E_2, \dots, E_n$ are independent.

Poisson process – counting hits

Claim (more-or-less from the book):

For a Poisson process with rate λ and a positive real number t, let N denote the number of hits in the interval $[0, t]$. Then N is a Poisson random variable with parameter λt.

Let's go over the proof on the board.

Monday’s draft problem

To be presented by Monday’s draftee.

The probability of getting dealt a full house in one hand of poker is approximately 0.0014. Use a Poisson approximation to approximate the probability that in 1000 hands of poker, you are dealt at least 2 full houses.

Sums of random variables – definition

Definition:

Let $X_1, X_2, \dots, X_n$ be several random variables, all on the *same* state space S. That is, each X_i is a function

$$X_i : S \to \mathbb{R}$$

Then the *sum* of $X_1, X_2, \dots, X_n$ is the random variable denoted $X_1 + \cdots + X_n$ and defined as

$$X_1 + \cdots X_n : S \to \mathbb{R}$$
$$s \mapsto X_1(s) + \cdots + X_n(s)$$

Linearity of expectation

Corollary 9.2:

Let $X_1, X_2, \dots, X_n$ be several random variables, all on the *same* state space S. Then

$$E[X_1 + \cdots + X_n] = E[X_1] + \cdots + E[X_n]$$

In fact, if $a_1, a_2, \dots, a_n$ are any real numbers, then

$$E[a_1X_1 + \cdots + a_nX_n] = a_1E[X_1] + \cdots + a_nE[X_n]$$

This is true in general, but for now, we will prove it assuming that our random variables are discrete. This makes things easier because $E[X] = \sum_{s\in S} X(s)p(s)$.