Lecture 9.1

Continuous Random Variables:

definition, expectation, variance, and simplest example

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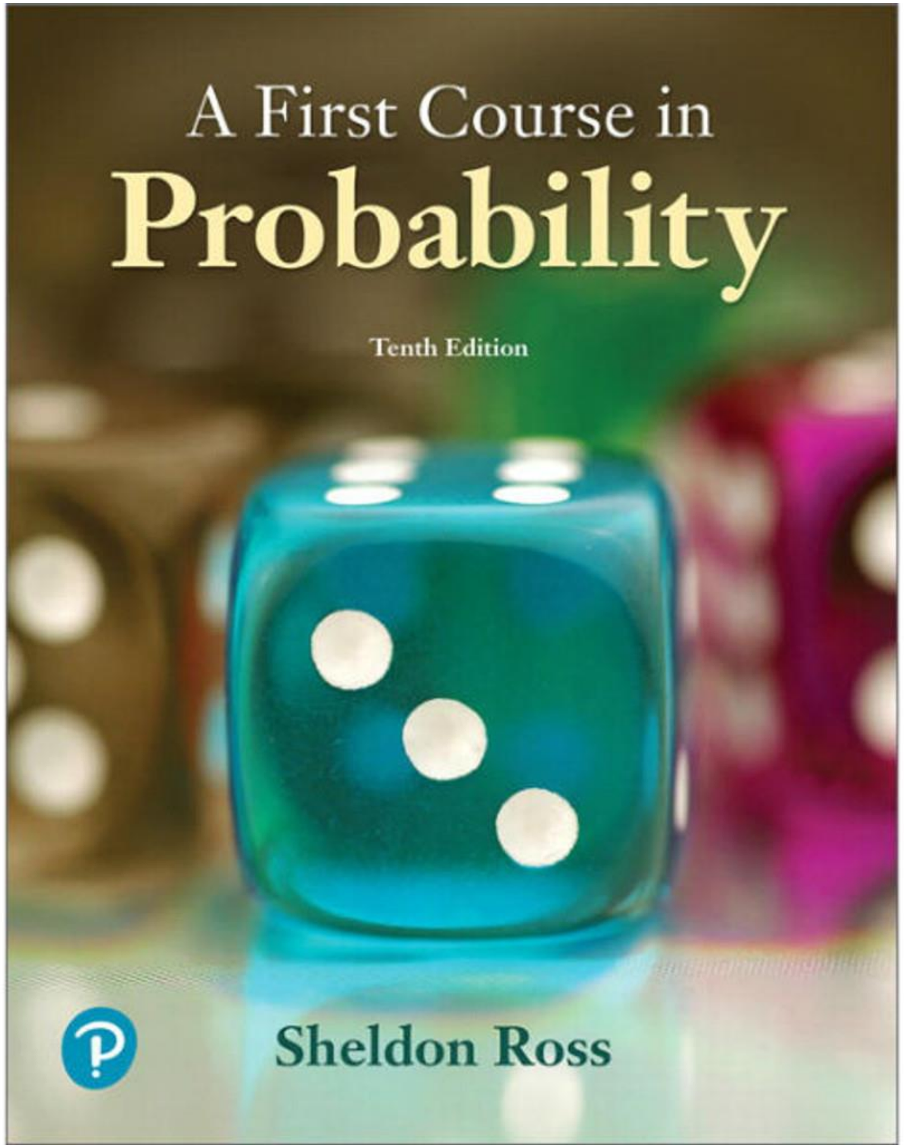

Today's reading: 5.1, 5.2, 5.3

Next class: 5.4

HW7 now available. Due Friday

Monday’s draft problem

To be presented by Monday’s draftee.

The probability of getting dealt a full house in one hand of poker is approximately 0.0014. Use a Poisson approximation to approximate the probability that in 1000 hands of poker, you are dealt at least 2 full houses.

Continuous Random Variables

Definition (more-or-less from the book):

A $\mathbb{R}$ random variable X (on a sample space S with probability measure P…) is called *(absolutely) continuous* if there exists a nonnegative function $f: \mathbb{R} \to \mathbb{R}$, called the *probability density function* of X, such that for any (measurable*) set $B \subset \mathbb{R}$

$$P\{X \in B\} = \int_{x \in B} f(x)\, dx$$

In particular,

$$P\{a \leq X \leq b\} = \int_a^b f(x)\, dx \qquad \text{and} \qquad P\{X = a\} = \int_a^a f(x)\, dx = 0$$

CDF of Continuous Random Variables

Definition (more-or-less from the book):

For a continuous random variable X with PDF $f\colon \mathbb{R} \to \mathbb{R}$, the *cumulative distribution function (CDF)* of X is the function $F\colon \mathbb{R} \to \mathbb{R}$ given by

$$F(a) = P\{X \leq a\} = \int_{-\infty}^{a} f(x)\, dx$$

In particular, by the fundamental theorem of calculus,

$$\frac{d}{da} F(a) = f(a)$$

Continuous Random Variables – Expectation & Variance

Definition (from the book):

For a continuous random variable X with PDF $f: \mathbb{R} \to \mathbb{R}$, the expectation value of X is

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

If we write $\mu = E[X]$, then the *variance* of X is defined to be

$$Var(X) = E[(X-\mu)^2] = \int_{x\in\mathbb{R}} (x-\mu)^2\, f(x)\, dx$$

Properties of Expectation

Most of the properties of expectation and variance we established for discrete random variables generalize appropriately:

Proposition 2.1. If g is a function of X, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Corollary 2.1.

$$\mathrm{E}[aX + b] = aE[X] + b$$

Properties of Variance

$$Var(X) = E[X^2] - (E[X])^2$$

$$Var(aX + b) = a^2 Var(X)$$

Proving these properties

The proofs of most of these properties are basically the same as the analogous properties for discrete random variables. The only exception is

Proposition 2.1. If g is a function of X, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

To prove this proposition, it is helpful to use

Lemma 2.1. If Y is a random variable that never takes a negative value, then

$$E[Y] = \int_0^{\infty} P\{Y > y\}\, dy$$

Uniform random variables

Definition

Fix two real numbers $\alpha < \beta$. A continuous random variable X is called *uniform on the interval (α, β)* if the PDF is

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & if\ \alpha < x < \beta \\ 0, & else \end{cases}$$

On the board, let's compute: the graph of $f(x)$, the CDF of $f(x)$ and its graph, the expectation and the variance.

Wednesday's draft problem

To be presented by Wednesday's draftee.

The PDF of X is given by

$$f(x) = \begin{cases} a + bx^2, & 0 < x < 1 \\ 0, & else \end{cases}$$

If $E[X] = 3/5$, find a and b.