**MA/STAT 416 - Probability - Final Exam (Practice)**
**Section 005/006**

**NAME:**________________________________________________________

Please write your name on the line above and circle your section number. There are five problems on this exam, pluse one extra credit question. Each problem will be graded out of 10 points, although the extra credit question will only be grade out of 4 points. Do not expect to get full credit on a problem if you do not justify or show your work. You have 120 minutes to complete the exam.

No resources are allowed other than a pen or pencil. In addition to this cover page, there is an extra blank sheet of scratch paper on the back. If you have any issues (missing pages, typos, unusual notation or definition, *etc.*), please raise your hand to let me know.

**Official Final Exam "Cheat sheet"**

$$n! \stackrel{\text{def}}{=} n(n-1)\cdots(2)(1) \qquad \binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} \qquad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

If $n_1 + n_2 + \cdots + n_r = n$, then

$$(x_1 + x_2 \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where

$$\binom{n}{n_1, n_2, \ldots, n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

A **probability measure** $P$ on a sample space $S$ is a function that assigns to each event $E \subset S$ a real number $P(E) \in \mathbb{R}$ subject to the following requirements:
**Axiom 1.** $0 \leq P(E) \leq 1$
**Axiom 2.** $P(S) = 1$
**Axiom 3.** For any sequence of mutually exclusive events $E_1, E_2, \ldots,$ in $S$,

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

**Inclusion-exclusion identity:** for any finite sequence of events $E_1, E_2, \ldots, E_n$ in a sample space $S$ with probability measure $P$, we have

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cdots E_{i_r}).$$

If $P(F) > 0$, the **conditional probability** of an event $E$ given $F$ is

$$P(E|F) \stackrel{\text{def}}{=} \frac{P(EF)}{P(F)}.$$

Two events $E$ and $F$ are **independent** if $P(EF) = P(E)P(F)$.

**Bayes Theorem (easy version):** For any events $E, H \subset S$, $P(H|E) = \frac{P(E|H)P(H)}{P(E)}$.

**Law of Total Probability:** If $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive events on a sample space $S$, then for any event $E \subset S$ we have $P(E) = \sum_{i=1}^{n} P(E|H_i)P(H_i)$.

A **($\mathbb{R}$-valued) random variable** is a function $X : S \to \mathbb{R}$, where $S$ is some sample space. A random variable is *discrete* if it takes on countably many values. A random variable is *(absolutely) continuous* if there exists a function $f : \mathbb{R} \to \mathbb{R}$ (called the probability density function of $X$) such that $P\{X < a\} = \int_{-\infty}^{a} f(x)dx$.

The **expectation value of a discrete random variable** $X$ is $E[X] = \sum_{x:p(x)>0} xp(x)$ (where $p(x)$ is the probability mass function of $X$). The **expectation value of a continuous random variable** $X$ is $E[X] = \int_{-\infty}^{\infty} xf(x)dx$.

The $n^{\mathbf{th}}$ **moment** of a random variable $X$ (either discrete or continuous) is $E[X^n]$. The **variance** of $X$ is $E[(X - E[X])^2] = E[X^2] - E[X]^2$.

**Bernoulli random variable:** A random variable $X$ is *Bernoulli* with parameter $0 \le p \le 1$ if $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$. $E[X] = p, Var(X) = p(1-p)$.

**Binomial random variable:** A random variable $X$ is *binomial* with parameters $n \in \{1, 2, 3, \dots\}$ and $0 \le p \le 1$ if for every integer $i$, $P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$. $E[X] = np$ and $Var(X) = np(1-p)$.

**Geometric random variable:** A random variable $X$ is *geometric* with parameter $0 \le p \le 1$ if for every positive integer $i$, $P\{X = i\} = p(1-p)^{i-1}$. $E[X] = 1/p$ and $Var(X) = (1-p)/p^2$.

**Poisson random variable:** A random variable $X$ is *Poisson* with parameter $\lambda > 0$ if for every non-negative integer $i$, $P\{X = i\} = e^{-\lambda}\lambda^i/i!$. $E[X] = \lambda$ and $Var(X) = \lambda$.

**Uniform random variable:** A random variable $X$ is *uniform* with parameters $a, b \in \mathbb{R}, a < b$ if it is continuous with PDF $f(x) = 1/(b-a)$ for $a < x < b$ (and $f(x) = 0$ otherwise). $E[X] = (a+b)/2$, $Var(X) = (b-a)^2/12$.

**Normal random variable:** A random variable $X$ is *normal* with parameters $\mu, \sigma^2 \in \mathbb{R}$ if it is continuous with PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

$E[X] = \mu, Var(X) = \sigma^2$.

**Exponential random variable:** A random variable $X$ is *exponential* with parameter $\lambda > 0$ if it is continuous with PDF $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ (and $f(x) = 0$ otherwise). $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$.

Two random variables $X$ and $Y$ are **independent** if for all subsets $A, B \subset \mathbb{R}$, $P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$.

If $X$ and $Y$ are jointly discrete random variables with joint PMF $p(x,y) = P\{X = x, Y = y\}$, then the **marginal PMF** for $X$ is $p_X(x) = \sum_{y:p(x,y)>0} p(x,y)$. The **conditional PMF** for $X$ conditioned on $Y$ is $p_{X|Y}(x) = \frac{p(x,y)}{p_Y(y)}$. Such jointly discretely distributed $X$ and $Y$ are indpendent if and only if $p(x,y) = p_X(x)p_Y(y)$.

If $X$ and $Y$ are jointly continuously distributed with joint PDF $f(x,y)$, then the **marginal PDF** for $X$ is $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$. The **conditional PDF** of $X$ conditioned on $Y$ is $f_{X|Y}(x) = \frac{f(x,y)}{f_Y(y)}$. Such jointly continuously distributed $X$ and $Y$ are independent if and only if $f(x,y) = f_X(x)f_Y(y)$.

The **covariance** of random variables $X$ and $Y$ is

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y].$$

| Q | 1 | 2 | 3 | 4 | 5 | 6 | TOTAL |
|---|---|---|---|---|---|---|---|
| Points | | | | | | | |

1. (a) [5 points] Consider two boxes, the first one containing 1 black and 1 white marble, and the second one containing 2 black and 1 white marbles. A box is selected at random, and a marble is drawn at random. What is the probability that the first box was selected given that the marble is white?

(b) [5 points] Ash Ketchum repeatedly has encounters with Pokémon, of which there are $N$ types. Each encouter is with a Pokémon of type $i$ with probabilty $p_i$, $i = 1, \ldots, N$. Suppose that it is Ash's $k^{\text{th}}$ encounter. What is the probability that it is with a new type of Pokémon that he has not seen before?

2. (a) [5 points] The number of people entering the elevator lobby of a large building is a Poisson random variable with $\lambda = 2$, *i.e.* at a rate of 2 people per minute. What is the probabilty that at least 9 people enter in the next 3 minutes?

(b) [5 points] Let $X$ be a continuous random variable with CDF $F$ and let $Y = F(X)$. Show that $Y$ is uniform on $(0, 1)$.

3. Consider two random variables $X$ and $Y$ with joint PDF $f(x, y) = 1/x, 0 < y < x < 1$.
(a) [8 pts] Compute the marginals $f_X$ and $f_Y$.

(b) [2 pts] Are $X$ and $Y$ independent? Explain.

4. [10 pts] How many times would you expect to roll a fair die until all 6 sides appeared at least once?

5. [10 pts] Let $X_1$ and $X_2$ be independent and identically distributed exponential random variables with the same parameter $\lambda > 0$. Find the PDF of the random variable $\frac{X_1}{X_1+X_2}$.

6. (a) [2 pts Extra Credit] Something about moment generating functions.

(b) [2 pts Extra Credit] Something about the Central Limit Theorem.