1(a) B_1: 1b, 1w , B_2: 2b, 1w

$$P(B_1 \mid w) = \frac{P(B_1 \cap w)}{P(w)}$$

$$\overset{\text{Bayes}}{=} \frac{P(w \mid B_1)\,P(B_1)}{P(w)} \underset{\text{LOTP}}{=} \frac{P(w \mid B_1)\,P(B_1)}{P(w \mid B_1)P(B_1) + P(w \mid B_2)P(B_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}} = \cdots =$$

✓

1(b) $P\{\text{new type in encounter } k\}$

$$\overset{LOTP}{=} \sum_{i=1}^{N} P\left\{\text{type } i \text{ in encounter } k \,\middle|\, \begin{matrix}\text{no type } i \text{ in first}\\ k-1 \text{ encounters}\end{matrix}\right\} \cdot P\left\{\begin{matrix}\text{no type } i \text{ in first}\\ k-1 \text{ encounters}\end{matrix}\right\}$$

$$= \sum_{i=1}^{N} P\{\text{type } i \text{ in } k\} P\{\text{no } i \text{ in first } k-1\}$$

$$= \sum_{i=1}^{N} p_i (1-p_i)^{k-1}$$

$$P\{\text{type } i \text{ in } k \mid \text{no } i \text{ in first } k-1\}$$

$$= \frac{P\{\text{"type } i \text{ in } k\text{" AND "no } i \text{ in first } k-1\text{"}\}}{P\{\text{no } i \text{ in first } k-1\}}$$

$$\overset{\text{independence}}{=} \frac{P\{\text{type } i \text{ in } k\} \cancel{P\{\text{no } i \text{ in first } k-1\}}}{\cancel{P\{\text{no } i \text{ in first } k-1\}}}$$

2(a) $X :=$ number of people in lobby after 3 minutes.

X is Poisson w/ rate $\lambda = 2 \cdot 3$

$$P(X \geq 9) = 1 - P(X \leq 8)$$

$$= 1 - \sum_{i=0}^{8} P(X = i)$$

$$= 1 - \sum_{i=0}^{8} \frac{e^{-6} 6^{i}}{i!}$$

2(b) $F(a) = CDF_X(a)$

$Y = F(X)$. Need to show

$$PDF_Y(a) = \begin{cases} 1 & \text{if } 0 \le a \le 1 \\ 0 & \text{else} \end{cases}$$

Let $\underbrace{0 < a < 1.}$ Then $PDF_Y(a) = \frac{d}{da} CDF_Y(a)$

$$= \frac{d}{da}\left[P\{Y \le a\}\right] = \frac{d}{da}\left[P\{F(X) \le a\}\right]$$

$$= \frac{d}{da}\left[P\{X \le F^{-1}(a)\}\right] = \frac{d}{da}\left[CDF_X(F^{-1}(a))\right]$$

$$= \frac{d}{da}\left[F(F^{-1}(a))\right] = \frac{da}{da} = 1. \checkmark$$

If $a < 0$ or $a > 1$, then Y "never takes that value." More precisely: because

$$\int_0^1 PDF_Y(a)\,da = \int_0^1 1\,da = 1$$

$PDF_Y(a) = 0$ for a NOT b/w 0 and 1. ✓

3.

$$f(x,y) = \begin{cases} 1/x & \text{if } 0<y<x<1 \\ 0 & \text{else.} \end{cases}$$

(a) $f_X(a) = \int_{-\infty}^{+\infty} f(a,y)\,dy \overset{\text{if } 0<a<1}{=} \int_0^a \frac{1}{a}\,dy$

$= \frac{1}{a}\int_0^a dy = \frac{1}{a}\left[y\right]_0^a = \frac{a-0}{a} = 1.$

$f_Y(b) = \int_{-\infty}^{+\infty} f(x,b)\,dx \overset{\text{if } 0<b<1}{=} \int_b^1 \frac{1}{x}\,dx$

$= \ln x\Big]_b^1 = \ln 1 - \ln b = -\ln b.$

If a not b/w $0,1$, then $f_X(a) = 0.$
$\quad\ b$ " " " " $\quad f_Y(b) = 0.$

(b) No, X, Y NOT independent! Suffices to pick a, b such that $f(a,b) \neq f_X(a)\,f_Y(b).$

Let $a = \frac{1}{4}$, $b = \frac{1}{2}$. Then $f(a,b) = 0.$
However,

$$f_X\left(\tfrac{1}{4}\right) f_Y\left(\tfrac{1}{2}\right) = 1\cdot\left(-\ln\tfrac{1}{2}\right) \neq \underset{f(a,b)}{\underset{\parallel}{0}}.$$

4. X: number of rolls needed to see all 6 sides of a die

E.g.: $\underbrace{6}_{X_1}\ \underbrace{1}_{X_2}\ \underbrace{6\ 2}_{X_3}\ \underbrace{6\ 5}_{X_4}\ \underbrace{2\ 5\ 3}_{X_5}\ \underbrace{4}_{X_6}$

$X = 10 = 1 + 1 + 2 + 2 + 3 + 1$

$X_i :=$ given that we've see $i-1$ different types of sides, how many rolls until we see a new side

$X = X_1 + X_2 + X_3 + X_4 + X_5 + X_6.$

Each X_i is geometric w/ success probability $(6-(i-1))/6$.

$$\Rightarrow E[X_i] = \frac{6}{6-(i-1)} = \frac{6}{6-i+1} = \frac{6}{7-i}.$$

$$\Rightarrow E[X] = \sum_{i=1}^{6} E[X_i] = \frac{6}{\cancel{7}-1} + \frac{6}{\cancel{7}-2} + \frac{6}{\cancel{7}-3} + \frac{6}{\cancel{7}-4} + \frac{6}{\cancel{7}-5} + \frac{6}{7-6}$$

$$= \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}.$$