MA/STAT 416 - Probability - Midterm Exam 1
Section 005/006

NAME: Solution Key

Please write your name on the line above. There are four problems on this exam and it is 50 minutes long. Each problem will be graded out of 10 points. Do not expect to get full credit on a problem if you do not justify or show your work.

No resources are allowed other than a pen or pencil. In addition to this cover page, there is an extra blank sheet of scratch paper on the back. If you have any issues (missing pages, typos, unusual notation or definition, *etc.*), please raise your hand to let me know.

Official Midterm 1 "Cheat sheet"

$n! \overset{\text{def}}{=} n(n-1)\cdots(2)(1)$ $\binom{n}{k} \overset{\text{def}}{=} \frac{n!}{k!(n-k)!}$ $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

If $n_1 + n_2 + \cdots + n_r = n$, then

$\binom{n}{n_1, n_2, \ldots, n_r} \overset{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$ and $(x_1 + x_2 \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$.

A **probability measure** P on a sample space S is a function that assigns to each event $E \subset S$ a real number $P(E) \in \mathbb{R}$ subject to the following requirements:

Axiom 1. $0 \le P(E) \le 1$

Axiom 2. $P(S) = 1$

Axiom 3. For any sequence of mutually exclusive events $E_1, E_2, \ldots$, in S,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-exclusion identity: for any finite sequence of events $E_1, E_2, \ldots, E_n$ in a sample space S with probability measure P, we have

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cdots E_{i_r}).$$

If $P(F) > 0$, the **conditional probability** of an event E given F is

$$P(E|F) \overset{\text{def}}{=} \frac{P(EF)}{P(F)}.$$

Two events E and F are **independent** if $P(EF) = P(E)P(F)$.

Q	points
1	10
2	10
3	10
4	10
TOTAL	40

A+

1. (a) [4 points] Carefully state the Law of Total Probability.

Given mutually exclusive and exhaustive events $F_1, \ldots, F_k$ in a sample space S, and another event $E \subset S$

$$P(E) = \sum_{i=1}^{k} P(E|F_i)\,P(F_i)$$

(b) [6 points] Now prove it.

$$\sum_{i=1}^{k} P(E|F_i)P(F_i) = \sum_{i=1}^{k} \frac{P(EF_i)}{P(F_i)} P(F_i)$$

$$= \sum_{i=1}^{k} P(EF_i).$$ Note that

$$\bigcup_{i=1}^{k} EF_i = E \cap \left(\bigcup_{i=1}^{k} F_i\right) \overset{\text{b/c } F_i \text{ exhaustive}}{=} E \cap S = E$$

and $$EF_i \cap EF_j = E(F_i \cap F_j) \underset{\text{b/c } F_i \text{ mutually exclusive}}{=} E\,\emptyset = \emptyset$$ for $i \neq j$.

Thus, by Axiom 3,

$$P(E) = P\left(\bigcup_{i=1}^{k} EF_i\right) \overset{\text{Axiom 3}}{=} \sum_{i=1}^{k} P(EF_i) = \sum_{i=1}^{k} P(E|F_i)P(F_i)$$

$\square$

2. There are 3 types of students: those who do their reading before class, those who do their reading after class, and those who don't do their reading. A student of the first type has a 90% probability of getting an A, a student of the second type has a 80% chance of getting an A, and a student of the third type has a 10% chance of getting an A. 60% of students are of the first type, 25% are of the second type and 15% are of the third type.

(a) [4 points] What is the probability that a random student gets an A?

By Law of Total Probability:

$$P(\text{get } A) = \sum_{i=1}^{3} P(\text{get } A \mid \text{student of type } i)\, P(\text{student of type } i)$$

$$= (.9)(.6) + (.8)(.25) + (.1)(.15) = .54 + .2 + .015$$
$$= .755$$

(b) [6 points] If a student gets an A, what is the probability that they are of the third type?

$$P(\text{type } i \mid \text{get } A) \underset{\text{Bayes}}{=} \frac{P(\text{get } A \mid \text{type } i)}{P(\text{get } A)} P(\text{type } i)$$

$i=3$:

$$P(\text{type } 3 \mid \text{get } A) = \frac{(.15)(.1)}{.755} = \frac{0.015}{0.755} \approx 2\%$$

3. (a) [4 points] What is the probability that a 5 card hand of poker is two pairs (that is, one pair, another pair with a different face value, and then a fifth card that doesn't have the same face value as either of the pairs)?

$$\#\{5 \text{ card hands that are 2 pairs}\} = \underbrace{\binom{13}{2}}_{\substack{\text{choose 2}\\\text{values for}\\\text{the 2 pairs}}} \times \underbrace{\binom{4}{2}}_{\substack{\text{choose 2}\\\text{suits for}\\\text{first pair}}} \times \underbrace{\binom{4}{2}}_{\substack{\text{choose 2}\\\text{suits for}\\\text{second pair}}} \times \underbrace{\binom{11}{1}}_{\substack{\text{choose}\\\text{fifth card's}\\\text{value}}} \times \underbrace{\binom{4}{1}}_{\substack{\text{fifth}\\\text{card's}\\\text{suit}}}$$

$$\Rightarrow P(2 \text{ pairs}) = \frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} \approx 4.75\%$$

(b) [6 points] What is the probability that in a 13 card hand of bridge, there is at least one suit that is not present?

Consider following events:

E: a 13 card hand has no hearts

F: " " " spades

G: " " " clubs

H: " " " diamonds.

We want $P(E \cup F \cup G \cup H) = \frac{\#(E \cup F \cup G \cup H)}{\binom{52}{13}}$. ~~To compute numerator,~~

Use inclusion-exclusion:

$$P(E \cup F \cup G \cup H) = \underbrace{P(E) + \cdots + P(H)}_{4 \text{ equal terms}} - \underbrace{P(EF) - \cdots - P(EH)}_{6 \text{ equal terms}} + \underbrace{P(EFG) + \cdots + P(FGH)}_{4 \text{ equal terms}} - P(EFGH)$$

$P(EFGH) = 0.$

$$P(EFG) = \binom{52 - 3 \cdot 13}{13} / \binom{52}{13} = 1 / \binom{52}{13}$$

$$P(EF) = \binom{52 - 2 \cdot 13}{13} / \binom{52}{13} = \binom{26}{13} / \binom{52}{13}$$

$$P(E) = \binom{52 - 1 \cdot 13}{13} / \binom{52}{13} = \binom{39}{13} / \binom{52}{13}$$

Cont. →

Therefore,

$$P(E \cup F \cup G \cup H)$$

$$= \frac{1}{\binom{52}{13}}\left[4\binom{39}{13} + 6\binom{26}{13} + 4\binom{13}{13} + 1 \cdot 0\right].$$

4. A and B play a series of games. Each game is independently won by A with probability p and by B with probability $1-p$. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of total wins is declared the winner of the series.

(a) [3 points] Find the probability that a total of 4 games are played.

Valid outcomes: ~~AAAB~~, ~~AABA~~, $\underbrace{\text{ABAA, BAAA}}_{\text{A wins after exactly 4 games}}$, ~~BBBA~~, ~~BBAB~~, $\underbrace{\text{BABB, ABBB}}_{\text{B wins after exactly 4 games}}$

$$P(\text{A or B wins after exactly 4 games}) = 2p^3(1-p) + 2p(1-p)^3.$$

(b) [7 points] Find the probabiliy that A is the winner of the series.

Note $$P(\text{A wins}) = \sum_{l=2}^{\infty} P(\text{A wins in exactly } l \text{ games})$$

$$= \sum_{l=2}^{\infty} P(\text{tied after } l-2 \text{ games})\,p^2 = p^2 \sum_{l=2}^{\infty} P(\text{tied after } l-2 \text{ games})$$

$$= p^2 \sum_{l=0}^{\infty} P(\text{tied after } l \text{ games}).$$ Note that

$$P(\text{tied after } l \text{ games}) = p\,P(\text{B up by 1 after } l-1 \text{ games}) + (1-p)\,P(\text{A up by 1 after } l-1 \text{ games})$$

$$P(\text{B up by 1 after } l-1 \text{ games}) = (1-p)\,P(\text{tied after } l-2 \text{ games})$$

$$P(\text{A up by 1 after } l-1 \text{ games}) = p\,P(\text{tied after } l-2 \text{ games}).$$

$$\Rightarrow P(\text{tied after } l \text{ games}) = 2p(1-p)\,P(\text{tied after } l-2 \text{ games}).$$

Note: $P(\text{tied after 0 games}) = 1 \Rightarrow P(\text{tied after } 2k \text{ games}) = [2p(1-p)]^k$

$P(\text{tied after 1 game}) = 0 \Rightarrow P(\text{tied after } 2k+1 \text{ games}) = 0.$

$$\Rightarrow \sum_{l=0}^{\infty} P(\text{tied after } l \text{ games}) = \sum_{k=0}^{\infty} P(\text{tied after } 2k \text{ games})$$

cont. →

$$= \sum_{k=0}^{\infty} [2p(1-p)]^r = \frac{1}{1-2p(1-p)}.$$

geometric series!

$$\Longrightarrow \boxed{P(A \text{ wins}) = \frac{p^2}{1-2p(1-p)}}.$$