MA/STAT 416 - Probability - Midterm Exam 2 (Practice) Section 005/006

NAME:

Please write your name on the line above and circle your section number (005 starts at 3:30 and 006 starts at 2:30). There are four problems on this exam and it is 50 minutes long. Each problem will be graded out of 10 points. Do not expect to get full credit on a problem if you do not justify or show your work.

No resources are allowed other than a pen or pencil. In addition to this cover page, there is an extra blank sheet of scratch paper on the back. If you have any issues (missing pages, typos, unusual notation or definition, etc.), please raise your hand to let me know.

Official Midterm 2 "Cheat sheet"

$$n! \stackrel{\text{def}}{=} n(n-1)\cdots(2)(1) \qquad \qquad \binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} \qquad (x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

If $n_1 + n_2 + \cdots + n_r = n$, then

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{n_1 + n_2 + \dots + n_r = n} {n \choose n_1, n_2, \dots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where

$$\binom{n}{n_1, n_2, \dots, n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

A probability measure P on a sample space S is a function that assigns to each event $E \subset S$ a real number $P(E) \in \mathbb{R}$ subject to the following requirements:

Axiom 1. $0 \le P(E) \le 1$

Axiom 2. P(S) = 1

Axiom 3. For any sequence of mutually exclusive events E_1, E_2, \ldots , in S,

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-exclusion identity: for any finite sequence of events E_1, E_2, \ldots, E_n in a sample space S with probability measure P, we have

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \dots < i_r} P(E_{i_1} \cdots E_{i_r}).$$

If P(F) > 0, the **conditional probability** of an event E given F is

$$P(E|F) \stackrel{\text{def}}{=} \frac{P(EF)}{P(F)}.$$

Two events E and F are **independent** if P(EF) = P(E)P(F).

Bayes Theorem (easy version): For any events $E, H \subset S$, $P(H|E) = \frac{P(E|H)P(H)}{P(E)}$. Law of Total Probability: If H_1, H_2, \dots, H_n are mutually exclusive and exhaustive events on a sample space S, then for any event $E \subset S$ we have $P(E) = \sum_{i=1}^{n} P(E|H_i)P(H_i)$.

A (\mathbb{R} -valued) random variable is a function $X:S\to\mathbb{R}$, where S is some sample space. A random variable is *discrete* if it takes on countably many values. A random variable is *(absolutely) continuous* if there exists a function $f:\mathbb{R}\to\mathbb{R}$ (called the probability density function of X) such that $P\{X< a\} = \int_{-\infty}^a f(x) dx$.

The expectation value of a discrete random variable X is $E[X] = \sum_{x:p(x)>0} xp(x)$ (where p(x) is the probability mass function of X). The expectation value of a continuous random variable X is $E[X] = \int_{-\infty}^{\infty} xf(x)dx$.

The n^{th} moment of a random variable X (either discrete or continuous) is $E[X^n]$. The variance of X is $E[(X - E[X])^2] = E[X^2] - E[X]^2$.

Bernoulli random variable: A random variable X is Bernoulli with parameter $0 \le p \le 1$ if $P\{X=1\}=p$ and $P\{X=0\}=1-p$. E[X]=p, Var(X)=p(1-p).

Binomial random variable: A random variable X is binomial with parameters $n \in \{1, 2, 3, ...\}$ and $0 \le p \le 1$ if for every integer i, $P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$. E[X] = np and Var(X) = np(1-p).

Geometric random variable: A random variable X is geometric with parameter $0 \le p \le 1$ if for every positive integer i, $P\{X = i\} = p(1 - p)^{i-1}$. E[X] = 1/p and $Var(X) = (1 - p)/p^2$.

Poisson random variable: A random variable X is *Poisson* with parameter $\lambda > 0$ if for every positive integer i, $P\{X = i\} = e^{-\lambda} \lambda^i / i!$. $E[X] = \lambda$ and $Var(X) = \lambda$.

Uniform random variable: A random variable X is uniform with parameters $a, b \in \mathbb{R}, a < b$ if it is continuous with PDF f(x) = 1/(b-a) for a < x < b (and f(x) = 0 otherwise). E[X] = (a+b)/2, $Var(X) = (b-a)^2/12$.

Normal random variable: A random variable X is normal with parameters $\mu, \sigma^2 \in \mathbb{R}$ if it is continuous with PDF

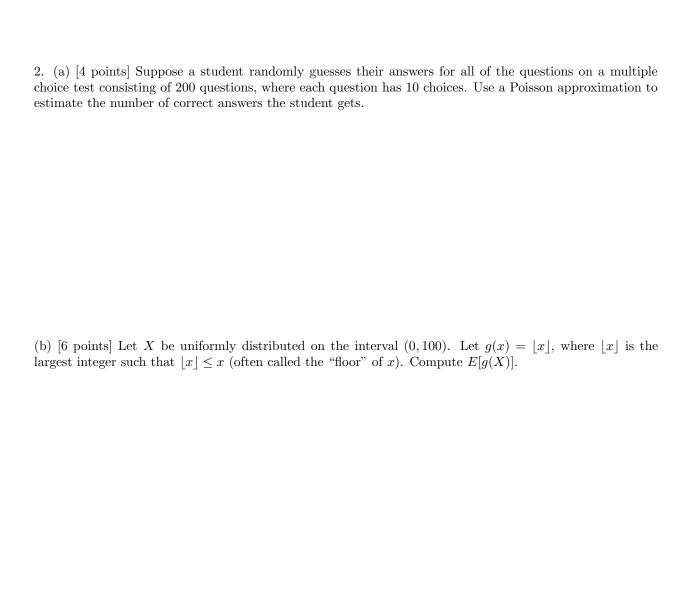
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}.$$

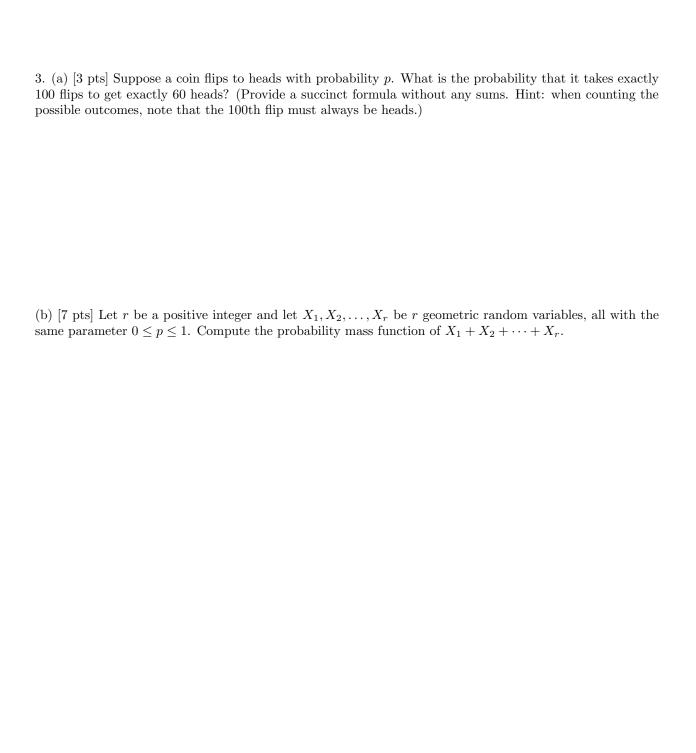
 $E[X] = \mu, Var(X) = \sigma^2.$

Exponential random variable: A random variable X is exponential with parameter $\lambda > 0$ if it is continuous with PDF $f(x) = \lambda e^{-\lambda x}$ for x > 0 (and f(x) = 0 otherwise). $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$.

Q	points
1	
2	
3	
4	
TOTAL	

1. (a) [5 points] Let X and Y be two discrete random variables. Prove that $E[X + Y] = E[X] + E[Y]$.
(b) [5 points] Let X be a continuous random variable with expectation $\mu = E[X]$. Prove that $E[(X - \mu)^2] = E[X^2] - E[X]^2$.





4	(a)	[4	ntsl	Recall	that a	nonnegative	random	variable	X	is	called	memorul	less	if
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$$P\{X > s + t | X > t\} = P\{X > s\}$$

for all s,t>0. Show directly from this definition that if X is memoryless and c>0, then cX is also memoryless.

(b) [6 pts] If X is an exponential random variable with parameter λ and c > 0, what kind of random variable is cX? Find the CDF of cX.