MA/STAT 416 - Probability - Midterm Exam 2 (Practice)
Section 005/006

NAME:____

Please write your name on the line above and circle your section number (005 starts at 3:30 and 006 starts at 2:30). There are four problems on this exam and it is 50 minutes long. Each problem will be graded out of 10 points. Do not expect to get full credit on a problem if you do not justify or show your work.

No resources are allowed other than a pen or pencil. In addition to this cover page, there is an extra blank sheet of scratch paper on the back. If you have any issues (missing pages, typos, unusual notation or definition, *etc.*), please raise your hand to let me know.

Official Midterm 2 "Cheat sheet"

$$n! \stackrel{\text{def}}{=} n(n-1)\cdots(2)(1) \qquad \binom{n}{k} \stackrel{\text{def}}{=} \frac{n!}{k!(n-k)!} \qquad (x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

If $n_1 + n_2 + \cdots + n_r = n$, then

$$(x_1 + x_2 \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n}\binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where

$$\binom{n}{n_1, n_2, \ldots, n_r} \stackrel{\text{def}}{=} \frac{n!}{n_1! n_2! \cdots n_r!}$$

A **probability measure** P on a sample space S is a function that assigns to each event $E \subset S$ a real number $P(E) \in \mathbb{R}$ subject to the following requirements:
Axiom 1. $0 \leq P(E) \leq 1$
Axiom 2. $P(S) = 1$
Axiom 3. For any sequence of mutually exclusive events $E_1, E_2, \ldots,$ in S,

$$P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$$

Inclusion-exclusion identity: for any finite sequence of events $E_1, E_2, \ldots, E_n$ in a sample space S with probability measure P, we have

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1 < \cdots < i_r} P(E_{i_1} \cdots E_{i_r}).$$

If $P(F) > 0$, the **conditional probability** of an event E given F is

$$P(E|F) \stackrel{\text{def}}{=} \frac{P(EF)}{P(F)}.$$

Two events E and F are **independent** if $P(EF) = P(E)P(F)$.

Bayes Theorem (easy version): For any events $E, H \subset S$, $P(H|E) = \frac{P(E|H)P(H)}{P(E)}$.

Law of Total Probability: If $H_1, H_2, \ldots, H_n$ are mutually exclusive and exhaustive events on a sample space S, then for any event $E \subset S$ we have $P(E) = \sum_{i=1}^{n} P(E|H_i)P(H_i)$.

A **($\mathbb{R}$-valued) random variable** is a function $X : S \to \mathbb{R}$, where S is some sample space. A random variable is *discrete* if it takes on countably many values. A random variable is *(absolutely) continuous* if there exists a function $f : \mathbb{R} \to \mathbb{R}$ (called the probability density function of X) such that $P\{X < a\} = \int_{-\infty}^{a} f(x)dx$.

The **expectation value of a discrete random variable** X is $E[X] = \sum_{x:p(x)>0} xp(x)$ (where $p(x)$ is the probability mass function of X). The **expectation value of a continuous random variable** X is $E[X] = \int_{-\infty}^{\infty} xf(x)dx$.

The $n^{\textbf{th}}$ **moment** of a random variable X (either discrete or continuous) is $E[X^n]$. The **variance** of X is $E[(X - E[X])^2] = E[X^2] - E[X]^2$.

Bernoulli random variable: A random variable X is *Bernoulli* with parameter $0 \leq p \leq 1$ if $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$. $E[X] = p, Var(X) = p(1 - p)$.

Binomial random variable: A random variable X is *binomial* with parameters $n \in \{1, 2, 3, \dots\}$ and $0 \leq p \leq 1$ if for every integer i, $P\{X = i\} = \binom{n}{i} p^i (1 - p)^{n-i}$. $E[X] = np$ and $Var(X) = np(1 - p)$.

Geometric random variable: A random variable X is *geometric* with parameter $0 \leq p \leq 1$ if for every positive integer i, $P\{X = i\} = p(1 - p)^{i-1}$. $E[X] = 1/p$ and $Var(X) = (1 - p)/p^2$.

Poisson random variable: A random variable X is *Poisson* with parameter $\lambda > 0$ if for every positive integer i, $P\{X = i\} = e^{-\lambda}\lambda^i/i!$. $E[X] = \lambda$ and $Var(X) = \lambda$.

Uniform random variable: A random variable X is *uniform* with parameters $a, b \in \mathbb{R}, a < b$ if it is continuous with PDF $f(x) = 1/(b - a)$ for $a < x < b$ (and $f(x) = 0$ otherwise). $E[X] = (a + b)/2$, $Var(X) = (b - a)^2/12$.

Normal random variable: A random variable X is *normal* with parameters $\mu, \sigma^2 \in \mathbb{R}$ if it is continuous with PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

$E[X] = \mu, Var(X) = \sigma^2$.

Exponential random variable: A random variable X is *exponential* with parameter $\lambda > 0$ if it is continuous with PDF $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ (and $f(x) = 0$ otherwise). $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$.

Q	points
1	
2	
3	
4	
TOTAL	

1. (a) [5 points] Let X and Y be two discrete random variables. Prove that $E[X+Y] = E[X] + E[Y]$.

(b) [5 points] Let X be a continuous random variable with expectation $\mu = E[X]$. Prove that $E[(X-\mu)^2] = E[X^2] - E[X]^2$.

2. (a) [4 points] Suppose a student randomly guesses their answers for all of the questions on a multiple choice test consisting of 200 questions, where each question has 10 choices. Use a Poisson approximation to estimate the number of correct answers the student gets.

(b) [6 points] Let X be uniformly distributed on the interval $(0, 100)$. Let $g(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer such that $\lfloor x \rfloor \leq x$ (often called the "floor" of x). Compute $E[g(X)]$.

3. (a) [3 pts] Suppose a coin flips to heads with probability p. What is the probability that it takes exactly 100 flips to get exactly 60 heads? (Provide a succinct formula without any sums. Hint: when counting the possible outcomes, note that the 100th flip must always be heads.)

(b) [7 pts] Let r be a positive integer and let $X_1, X_2, \ldots, X_r$ be r geometric random variables, all with the same parameter $0 \leq p \leq 1$. Compute the probability mass function of $X_1 + X_2 + \cdots + X_r$.

4. (a) [4 pts] Recall that a nonnegative random variable X is called *memoryless* if

$$P\{X > s + t | X > t\} = P\{X > s\}$$

for all $s, t > 0$. Show directly from this definition that if X is memoryless and $c > 0$, then cX is also memoryless.

(b) [6 pts] If X is an exponential random variable with parameter λ and $c > 0$, what kind of random variable is cX? Find the CDF of cX.