1. (a) [5 points] Let X and Y be two discrete random variables. Prove that $E[X+Y] = E[X] + E[Y]$.

Too long! See Ch 4: Cor 9.2, Prop 9.1

(b) [5 points] Let X be a continuous random variable with expectation $\mu = E[X]$. Prove that $E[(X-\mu)^2] = E[X^2] - E[X]^2$.

$$\mu = E[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$$ where $f(x)$ is PDF of X.

$$E[(X-\mu)^2] = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{+\infty} (x^2 - 2\mu x + \mu^2) f(x)\, dx$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\, dx - 2\mu \int_{-\infty}^{+\infty} x\, f(x)\, dx + \mu^2 \int_{-\infty}^{+\infty} f(x)\, dx$$

$$= E[x^2] - 2\mu E[X] + \mu^2 \cdot 1$$

$$= E[x^2] - 2\mu^2 + \mu^2 = E[x^2] - \mu^2 = E[x^2] - E[x]^2$$

2. (a) [4 points] Suppose a student randomly guesses their answers for all of the questions on a multiple choice test consisting of 200 questions, where each question has 10 choices. Use a Poisson approximation to estimate the number of correct answers the student gets.

$X = \#\{\text{correct answers}\}$ is binomial w/ parameters $n = 200$ and $p = 1/10$. Let $\lambda = np$ and Y be Poisson w/ parameter λ. Then, for all ~~post~~ nonnegative integers $k \leq 200$

$$P\{X=k\} \approx P\{Y=k\} = e^{-np}\frac{(np)^k}{k!}.$$

~~The expected~~ number is np - - -

(b) [6 points] Let X be uniformly distributed on the interval $(0, 100)$. Let $g(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer such that $\lfloor x \rfloor \leq x$ (often called the "floor" of x). Compute $E[g(X)]$.

Let's find PMF of $g(X)$ first. Note $g(X)$ only takes values $i = 0, 1, 2, \ldots, 99$. Pick such an i:

$$p(i) = P\{\lfloor X \rfloor = i\} = P\{i \leq X < i+1\} = \frac{i+1-i}{100-0}$$

$$= \frac{1}{100}.$$ Using this:

$$E[g(X)] = \sum_{i=0}^{99} i\,p(i) = \frac{1}{100}\sum_{i=0}^{99} i$$

$$= \frac{1}{\cancel{100}}\left(\frac{99 \cdot \cancel{100}}{2}\right) = 49.5.$$

3. (a) [3 pts] Suppose a coin flips to heads with probability p. What is the probability that it takes exactly 100 flips to get exactly 60 heads? (Provide a succinct formula without any sums. Hint: when counting the possible outcomes, note that the 100th flip must always be heads.)

Flip 100 is H. Must choose where among first 99 flips the first 59 H's occur. All choices are equally likely.

$$\binom{99}{59} p^{59}(1-p)^{99-59} p = \binom{99}{59} p^{60}(1-p)^{40}$$

(b) [7 pts] Let r be a positive integer and let $X_1, X_2, \ldots, X_r$ be r independent geometric random variables, all with the same parameter $0 \leq p \leq 1$. Compute the probability mass function of $X_1 + X_2 + \cdots + X_r$.

$P\{X_1 + \cdots + X_r = k\} = P\{Y = k\}$ where Y is the random variable that counts how many flips we need of a single coin (w/ prob(H) $= p$) until we get exactly r heads. Using this:

$$P\{Y = k\} = \binom{k-1}{r-1} p^{r-1}(1-p)^{k-r} p.$$

4. (a) [4 pts] Recall that a nonnegative random variable X is called *memoryless* if

$$P\{X > s + t | X > t\} = P\{X > s\}$$

for all $s, t > 0$. Show directly from this definition that if X is memoryless and $c > 0$, then cX is also memoryless.

Let $Y = cX$. Need to show Y satisfies

$P\{Y > s+t \mid Y > t\} = P\{Y > s\}$ for all $s,t>0$. Let $s,t>0$.

$$P\{Y > s+t \mid Y > t\} = P\{cX > s+t \mid cX > t\}$$

$$= P\{X > \tfrac{s}{c} + \tfrac{t}{c} \mid X > \tfrac{t}{c}\} \underset{X \text{ memoryless}}{=} P\{X > \tfrac{s}{c}\}$$

$$= P\{cX > s\} = P\{Y > s\}.$$

(b) [6 pts] If X is an exponential random variable with parameter λ and $c > 0$, what kind of random variable is cX? Find the CDF of cX.

cX is exponential. This is b/c X exponential $\Rightarrow$ X memoryless $\underset{(a)}{\Rightarrow}$ cX memoryless $\Rightarrow$ cX exponential

(Since "continuous + positive + memoryless" $\Leftrightarrow$ "continuous + positive + exponential")

$$CDF_{cX}(a) = P\{cX \le a\} = P\{X \le \tfrac{a}{c}\}$$

$$= \int_{-\infty}^{a/c} \lambda e^{-\lambda x}\, dx = \cdots = 1 - e^{-\lambda a/c}.$$