# MATH 562 - Intro to Differential Geometry & Topology

Following Lee's "Introduction to Smooth Manifolds", $2^{nd}$ edition, GTM 218, 2012, during first portion of course. Chapters 1-17.

## Lecture 1.1

**Reading:** front matter, Appendix A, Ch. 1 pp. 1-9.

**Practice Problems:** Exercises A.1, A.2, A.3, A.10, A.11, A.15, A.21, A.46, Ex 1.1

**Homework:** Ex 1.6 + 1.7, Prob. 1-1.

### Some point-set topology review

**Def** A topological space $(X, \mathcal{T})$ is a set $X$ together with a set of subsets $\subseteq 2^X$ satisfying:

(i) $X, \emptyset \in \mathcal{T}$

(ii) the union of any family of sets in $\mathcal{T}$ is again in $\mathcal{T}$

(iii) the intersection of any finite family of sets in $\mathcal{T}$ is again in $\mathcal{T}$.

The elements of $\mathcal{T}$ are called the open subsets of the topological space.

Rmks (i) When $\mathcal{T}$ is understood, often just write $X$ instead of $(X, \mathcal{T})$.

(ii) $X$ and $\mathcal{T}$ are usually uncountably infinite.

(iii) often say "space" or "topology" instead of "topological space"

Def Let $X$ be a space and $p \in X$.

- a neighborhood of $p$ is an open set $U \in \mathcal{T}$ with $p \in U$. (Note: sometimes called open neighborhood)
- $S \subseteq X$ is closed if $X - S$ is open.
- More you should review if needed: interior, exterior, closure, boundary, ...

Def Let $X$ and $Y$ be topological spaces. A function $F : X \to Y$ is continuous if for every open subset $U \subseteq Y$, $F^{-1}(U)$ is an open subset of $X$.

Def $F : X \to Y$ is called a homeomorphism if it is continuous and invertible with a continuous inverse.

Ex $\mathbb{R}^n$ with the usual notion of open set generated by the metric

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sqrt{\sum_i (x_i - y_i)^2}.$$

Open sets are unions of open balls $B_r(x)$.

**Def** A basis of a topology $(X, \mathcal{T})$ is a subset $\mathcal{B} \subseteq \mathcal{T}$ such that every open set in $\mathcal{T}$ is a union of open sets in $\mathcal{B}$.

**Def** A topology $(X, \mathcal{T})$ is second-countable if it admits a countable basis.

**Ex A.15** $\mathbb{R}^n$ is second-countable. ☆ Ended 8/19.

**Def** A topological space $(X, \mathcal{T})$ is Hausdorff (or $T2$) if for every $p, q \in X$ distinct, there exists $U, V \in \mathcal{T}$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$.

**Ex A.10** $\mathbb{R}^n$ is Hausdorff. (Every metric space is.)

## Topological Manifolds

**Def** A topological space $M$ is called locally Euclidean of dimension $n$ if each point $p \in M$ has an open neighborhood that is homeomorphic to an open set in $\mathbb{R}^n$.

**Ex 1.1** Might as well assume every point has a neighborhood homeomorphic to an open ball in $\mathbb{R}^n$.

**Def** A topological manifold of dimension $n$

- The "second-countable" condition guarantees the manifolds are not "too big" for "boring" reasons. Indeed, Prop 1.11 shows that the connected components of a topological manifold are the same as the path components, and, more to the point, there are at most countably many components. Thus, a disjoint union of uncountably many copies of $\mathbb{R}$ is not a topological manifold.

Similarly, for connected spaces that are locally Euclidean and Hausdorff, "second-countable" is equivalent to "paracompact" (Prob 1-5). Thus, the "long line" is not a topological manifold.

Ex Let $X$ be an uncountable set. Fix a order on $X$. The long ray is the quotient of $\bigsqcup_{x \in X} [0,1]$ formed by identifying the $1$ in the $x^{th}$ interval with the $0$ in the $S(x)^{th}$ interval, where $S(x)$ is the

successor of $x$ with respect to the well-order.
To be more precise, the long ray is the topological space formed by the order topology on the lexicographic order of $\omega_1 \times [0,1]$, where $\omega_1$ is the first uncountable ordinals

The long line is formed by gluing two copies of the long ray together.

- Open subsets of an $n$-manifold are themselves manifolds.

- Note: topological manifolds are topological spaces that satisfy some conditions. There is no additional data beyond their topology. For smooth manifolds, we will need more structure called an atlas of smooth charts.