Lecture 1.2

Reading: Ch. 1 pp. 11-29

Practice Problems: Ex. 1.18, 1.20

Homework: Probs 1-7, 1-12.

Coordinate charts

Def Let M^n be a topological manifold. A (coordinate) chart is a pair (U, φ) where $U \subseteq M$ open and $\varphi: U \to \hat{U}$ is a homeomorphism to an open set $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The component functions x^i such that $\varphi(p) = (x^1(p), \ldots, x^n(p))$ are called local coordinates on U.

Examples

Ex 1.3 $U \subseteq \mathbb{R}^n$ open, $f: U \to \mathbb{R}^k$ cts. The graph of f is

$$\Gamma(f) = \{ (x, f(x)) \mid x \in U \} \subseteq \mathbb{R}^n \times \mathbb{R}^k.$$

Note that
$$\begin{aligned} \pi: \Gamma(f) &\to U \\ (x, f(x)) &\mapsto x \end{aligned}$$
is a homeomorphism.

Thus, $\Gamma(f)$ is a topological manifold.

Ex 1.4 Unit sphere in $\mathbb{R}^{n+1}$ is

$$S^n := \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}.$$

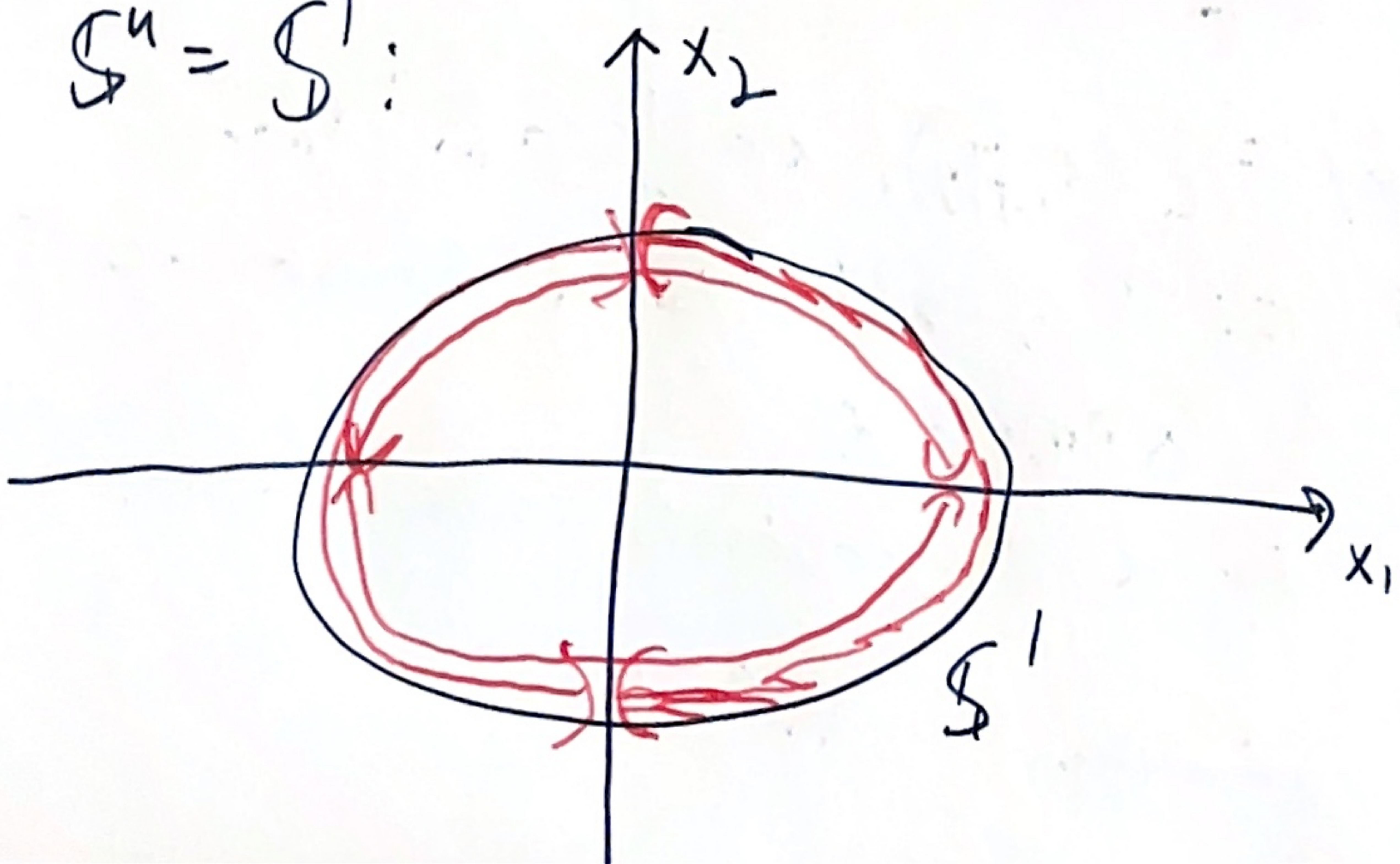
Since S^n is a subspace of $\mathbb{R}^{n+1}$, it is Hausdorff and second-countable. Let us show it is locally Euclidean. Define, for each $i = 1, \dots, n+1$, the open hemispherical caps

$$U_i^+ = \{ (x_1, \dots, x_{n+1}) \mid x_i > 0 \} \subseteq S^n$$
$$U_i^- = \{ (x_1, \dots, x_{n+1}) \mid x_i < 0 \} \subseteq S^n.$$

Can check that these $2(n+1)$ are an open cover of S^n. Moreover, each $U_i^\pm$ is homeomorphic to a open ball because it is a graph of a continuous function on the unit open ball

$$B^n := \{ x \in \mathbb{R}^n \mid \|x\| < 1 \} \subseteq \mathbb{R}^n.$$

E.g. when $n = 1$, have four coordinate charts on $S^n = S^1$:



Ex 1.5 $\mathbb{RP}^n$ defined as set of all lines through $0 \in \mathbb{R}^{n+1}$. We put "quotient topology" on $\mathbb{RP}^n$ via the map

$$\pi : \mathbb{R}^{n+1}-\{0\} \to \mathbb{RP}^n$$
$$x \mapsto \mathrm{Span}(x) =: [x].$$

by saying $U \subseteq \mathbb{RP}^n$ is open iff $\pi^{-1}(U) \subseteq \mathbb{R}^{n+1}-\{0\}$ is open. We claim that $\mathbb{RP}^n$ is a topological n-manifold. You'll check Hausdorff and second-countable on HW1. We will now show that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

First let's define an open cover. Let

$$\tilde{U}_i := \{x \in \mathbb{R}^n-\{0\} \mid x^i \neq 0\} \subseteq \mathbb{R}^n-\{0\}.$$

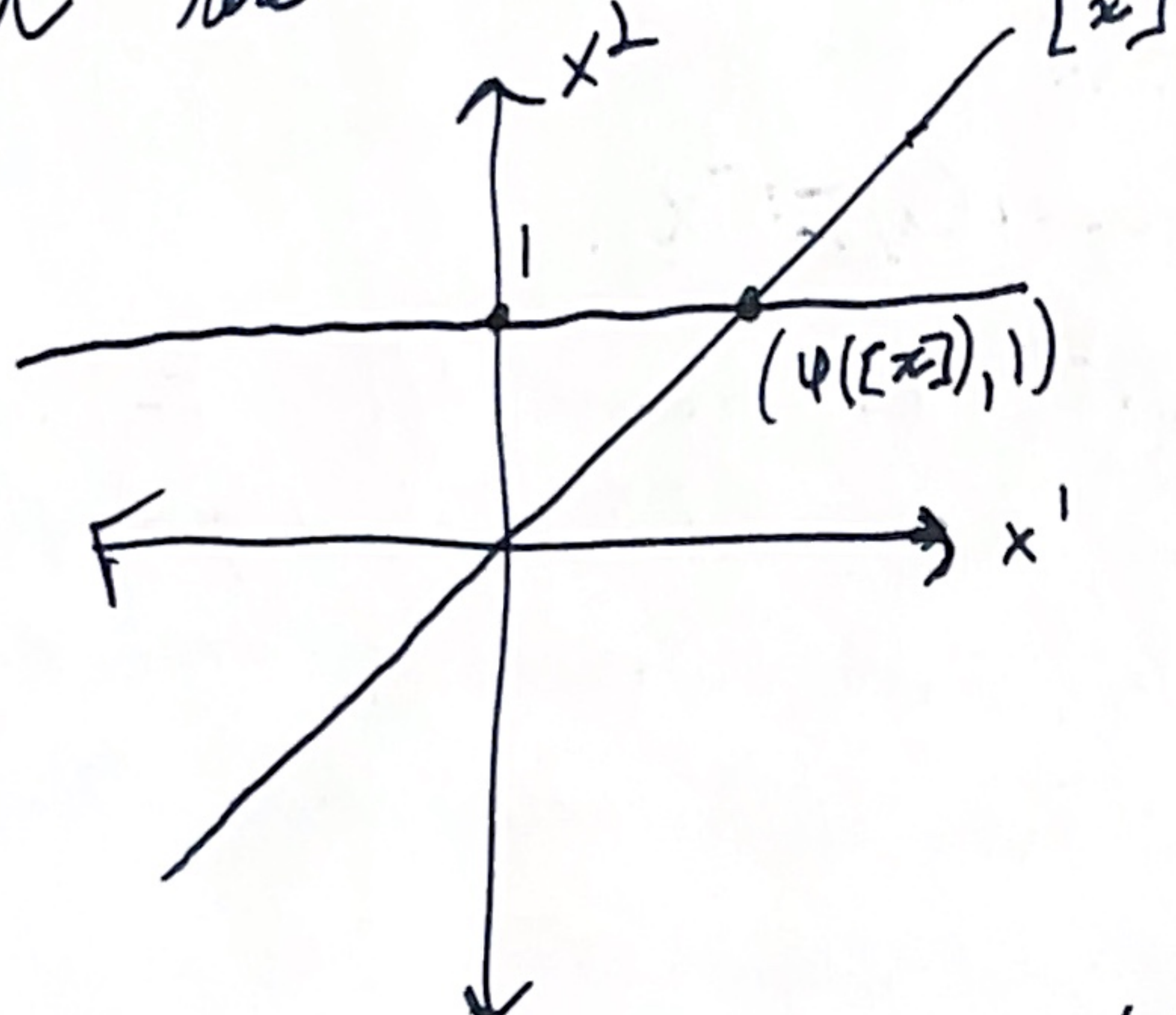
Then $\pi(\tilde{U}_i) =: U_i$ is open in $\mathbb{RP}^n$. Easy to check that $U_1, \dots, U_{n+1}$ forms an open cover of $\mathbb{RP}^n$.

We conclude by showing each U_i is homeomorphic to $\mathbb{R}^n$. Easy to check that

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n$$
$$[x] \mapsto \left(\frac{x^1}{x^i}, \frac{x^2}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^n}{x^i}\right)$$

a homeomorphism.

For example, when $n=1$ and $i=2$, U_i is all lines in $\mathbb{R}^2$ through 0 that aren't the line $x^2=0$.



Ex 1.8 If $M_1,\dots,M_k$ are topological manifolds of dimension $n_1,\dots,n_k$, then $M_1\times\dots\times M_k$ a manifold of dimension $n_1+\dots+n_k$. E.g. n-torus
$$\mathbb{T}^n := S^1\times\dots\times S^1.$$

Smooth structures and atlases

Def Let $U\subseteq\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$ be open. A function $F:U\to V$ is called <u>smooth</u> (or C^∞) if it has continuous partial derivatives of all order.

Def $F: U \to V$ is a <u>diffeomorphism</u> if it is a smooth, bijection with smooth inverse.

[<u>Q</u>: How necessary is it to assume inverse is smooth?]

Def Let M be a topological manifold. An <u>atlas</u> is an open cover of M together with charts on each set on the cover.

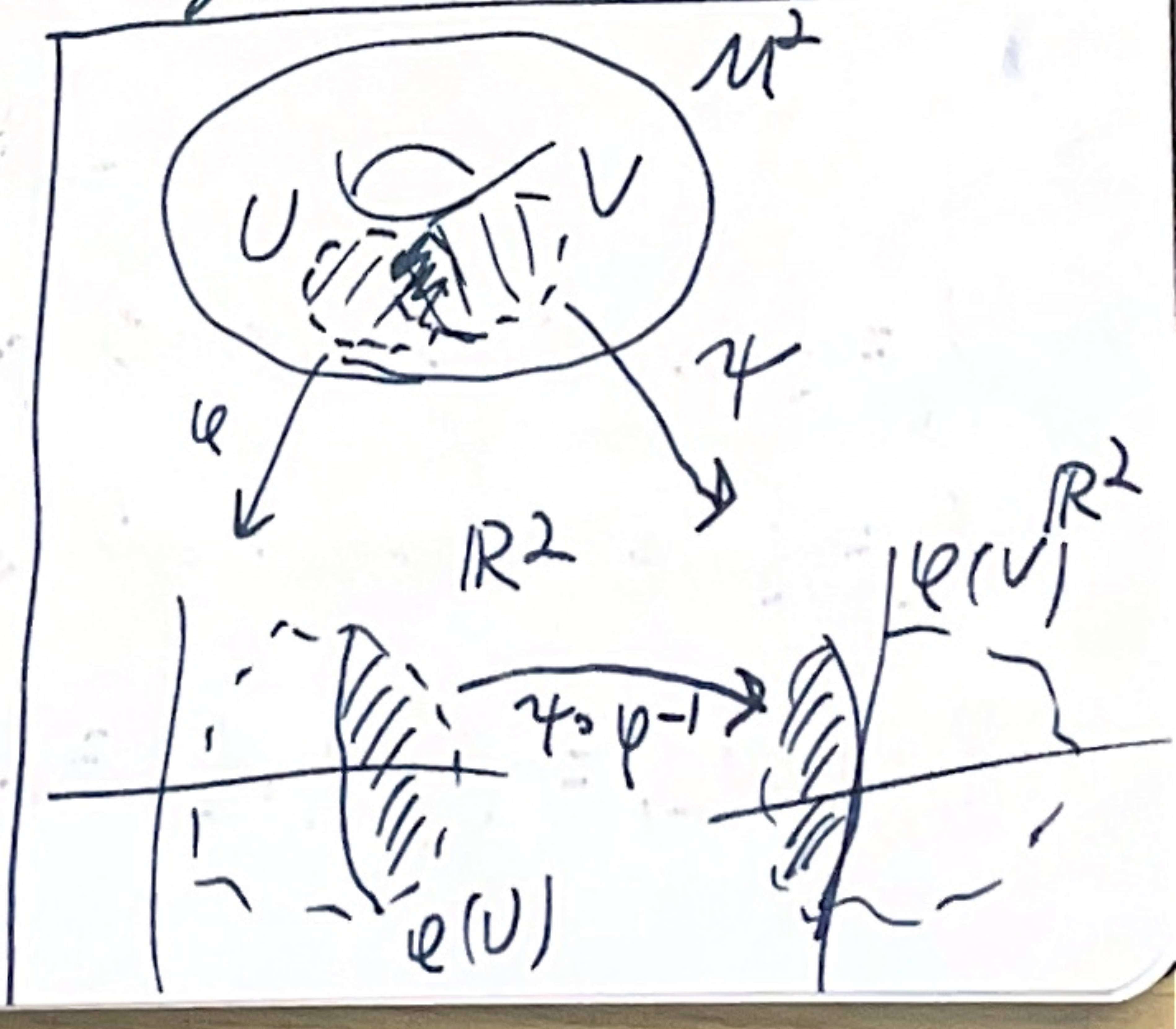
Def Let $(U, \varphi), (V, \psi)$ be two charts on M^n. The <u>transition map</u> from φ to ψ is the function

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

If $\psi \circ \varphi^{-1}$ is a smooth function, then φ and ψ are said to be <u>smoothly compatible</u>.

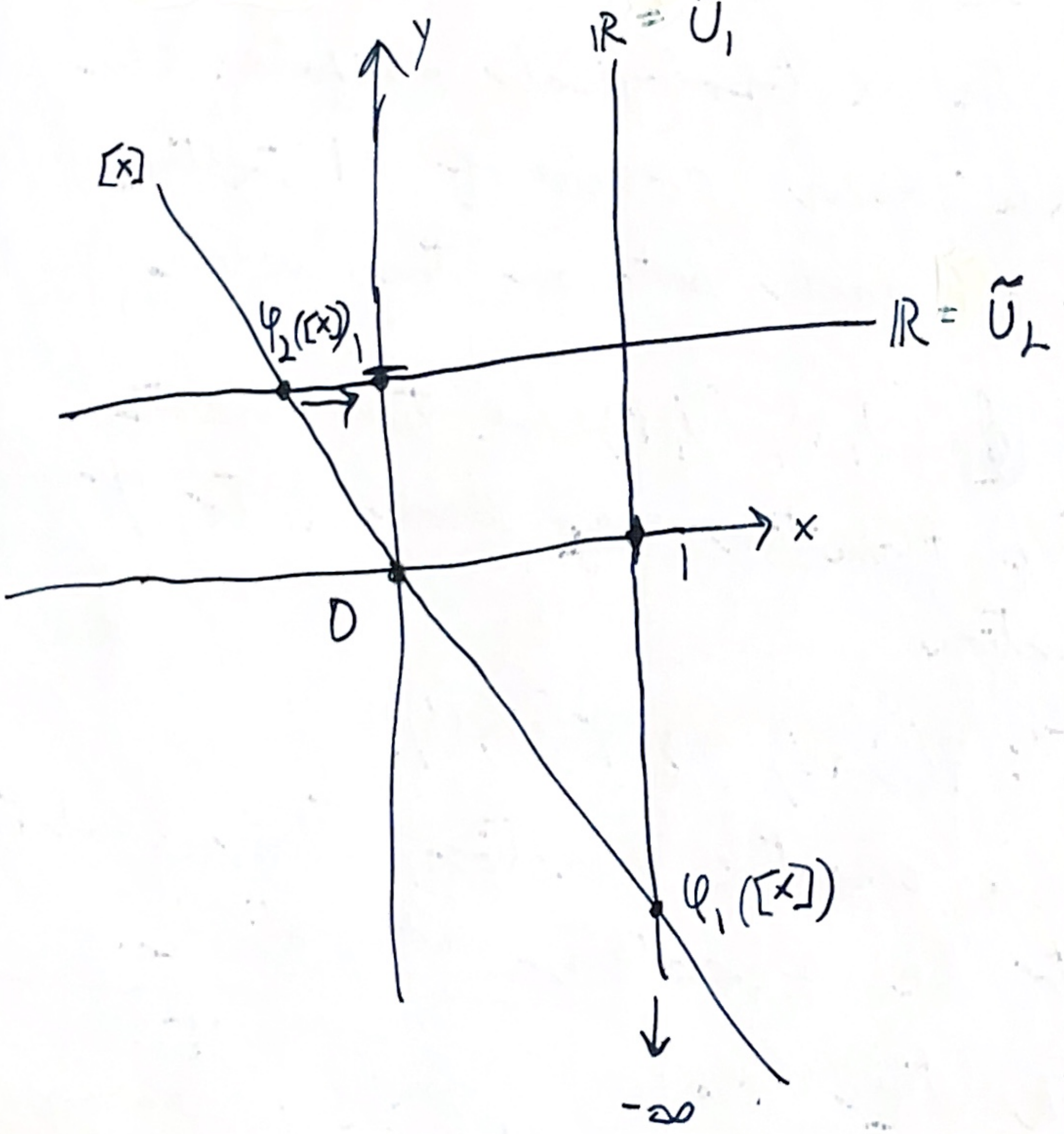
Def A <u>smooth atlas</u> on M is an atlas such that all charts are smoothly compatible.

<u>Ex 1.31</u> The atlas on S constructed in Ex 1.4 is smooth.



$\underline{\text{Ex 1.33}}$ - The atlas we constructed on $\mathbb{RP}^n$ is smooth. The transition functions are...

Warm-ups first!



$U_1 = \{\text{all lines except } y\text{-axis}\} \xrightarrow{\varphi_1} \tilde{U}_1 = \mathbb{R}$

$U_2 = \{\text{all lines except } x\text{-axis}\} \xrightarrow{\varphi_2} \tilde{U}_2 = \mathbb{R}$

$\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{R} - \{0\}$.

$\varphi_2 \circ \varphi_1^{-1} : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ Diffeomorphism? ✓

$u \mapsto 1/u$.

In general: for $i > j$, have

$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

$$[u] \mapsto \left(\frac{u^1}{u^j}, \dots, \frac{u^{j-1}}{u^j}, \frac{u^{j+1}}{u^j}, \dots, \frac{u^{i-1}}{u_j}, \frac{1}{u_j}, \frac{u^i}{u^j}, \dots, \frac{u^n}{u^j}\right)$$

...

Prop 1.17 Let M be a topological manifold.
(a) Every smooth atlas is contained in a unique maximal smooth atlas.
(b) Two smooth atlases determine the same maximal atlas iff their union is a smooth atlas.

Proof (b) follows from (a) immediately.
To prove (a), fix an atlas $\mathcal{A} = \{(U, \varphi)\}$. Let $\bar{\mathcal{A}}$ be the atlas consisting of all charts that are smoothly compatible with the charts in $\mathcal{A}$. We need to show ~~two~~ three things:
(i) $\bar{\mathcal{A}}$ is a smooth atlas, (ii) maximal, (iii) unique.

In fact, (ii) is immediate. Moreover, if (i) is true, then (iii) follows immediately.

To prove (i), let (U, φ), (V, ψ) be in $\bar{\mathcal{A}}$. We must check that they are smoothly compatible. It suffices to check that $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth at every $p \in \varphi(U \cap V)$ "individually." To this end, pick (W, θ) in $\mathcal{A}$ (not $\bar{\mathcal{A}}$) with $p \in W$. Then in a neighborhood of p, $\psi \circ \varphi^{-1}$ can be expressed as (namely, $\varphi(U \cap V \cap W)$) $\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$. Since $(\psi \circ \theta^{-1})$ and $(\theta \circ \varphi^{-1})$ are both smooth at p, we conclude that $\psi \circ \varphi^{-1}$ is too. □

Def Let be a topological manifold. A <u>smooth structure</u> on M is a maximal smooth atlas on M.

<u>Rmks</u>

- There exist topological manifolds with no smooth structure. [Kervaire, 1960]

- A topological manifold can have infinitely many distinct smooth structures. See Prob 1-6

- Even compact topological manifolds can! Milnor showed there are 28 distinct smooth structures on S^7. These are called exotic spheres.

- Open problem (Smooth 4-dim'l Poincaré conjecture): do there exist exotic smooth structures on S^4?