Lecture 10.1

Reading: First two section of Ch. 11

Practice: 11.2, 11.3, 11.5, 11.7, 11.10, 11.12

Homework: 10-15, 11-5, 11-16

Covectors

Def If V a vector space (over $\mathbb{R}$), the dual vector space is

$$V^* = \mathrm{Hom}(V, \mathbb{R}) = \{\lambda : V \to \mathbb{R} \text{ linear}\}.$$

Elements of V^* are covectors, or linear functionals.

Prop 10.11 If $(E_1, \dots, E_n)$ a basis of V, then the linear maps

$$\varepsilon^i : V \to \mathbb{R} \qquad i = 1, \dots, n$$

defined by $\varepsilon^i(E_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$ are a

basis of V^*, called dual basis to $(E_1, \dots, E_n)$.

Fact V is finite dimensional iff V^* finite dimensional. iff $V \cong V^*$. However, no natural isomorphism $V \to V^*$!

Def Given $A : V \to W$, A^* is the dual linear map (or transpose)

$A^* : W^* \to V^*$ defined by

$$A^*\omega(v) = \omega(Av) \qquad \forall \omega \in W^*, v \in V.$$

Prop 11.8 Given vector space V, define linear map

$$\xi : V \to V^{**}$$

by $\xi(v)(\omega) = \omega(v)$ for $\omega \in V^*$. If V finite dimensional

Then ξ is an isomorphism ("natural").

Proof Suffices to show ξ injective. Given $v \neq 0$, pick a basis $(E_1, \dots, E_n)$ with $E_1 = v$, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be the dual basis. Then $\underbrace{\varepsilon^1(v)}_{\text{"}\xi(v)(\varepsilon^1)\text{"}} = \varepsilon^1(E_1) = \delta^1_1 = 1 \neq 0$. ∎

Cotangent space and bundle

Def If M a smooth manifold and $p \in M$, the cotangent space at p is

$$T_p^* M := (T_p M)^*.$$

Elements of $T_p^* M$ are called tangent covectors at p.

Ex 11.10 If $E \to M$ a vector bundle, the dual bundle $E^* \to M$ has fibers $(E^*)_p := (E_p)^*$.

It has a ~~basis~~ smooth vector bundle structure with transition functions

$\tau^*(p) := (\tau(p)^{-1})^T$ for any transition function $\tau : U \to GL(k, \mathbb{R})$ of E.

Def The <u>cotangent bundle</u> of M is the dual vector bundle to $TM \to M$, denoted $T^*M \to M$.

Def A section of T^*M is called a <u>covector field</u> or a <u>differential 1-form</u>.

this will make sense later.

Space of all sections of T^*M denoted $\mathfrak{X}^*(M)$.

<u>Note</u>: Have a pairing

$$\mathfrak{X}^*(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

$$(\omega, X) \mapsto \{p \mapsto \omega_p(X_p)\}$$

<u>Prop 11.11</u> Let $\omega : M \to T^*M$ be a rough covector field. TFAE:

(a) ω is smooth

(b) In every coordinate chart, components of ω are smooth

(c) skip

(d) $\forall X \in \mathfrak{X}(M)$, $\omega(X)$ is a smooth ($\mathbb{R}$-valued) function on M

(e) skip

We can define coframes exactly as expected: they are just frames of the cotangent bundle.

Differentials of functions

Def Given $F: M \to \mathbb{R}$ smooth, the differential of F is the differential 1-form $dF \in \mathfrak{X}^*(M)$ defined by

$$dF_p(v) := vF \quad \text{for } v \in T_pM.$$

Prop 11.18 If F smooth, then dF is too.

Proof: Apply Prop 11.11 (d). $\square$

But wait! We already defined the notation dF: it is the total derivative!

$$dF : TM \to T\mathbb{R} \qquad \text{vs.} \qquad dF : M \to T^*M$$
$$(p,v) \mapsto (F(p), dF_p v) \qquad\qquad p \mapsto dF_p$$

Fortunately, these are essentially the same. Indeed, $dF: TM \to T\mathbb{R}$ yields, for each $p \in M$

$dF_p : T_pM \longrightarrow T_{F(p)}\mathbb{R} \cong \mathbb{R}$

canonical!

$\Rightarrow dF_p \in T_p^*M$

Lecture 10.2

Reading: Finish Ch 11

Practice: 11.32, 11.35, 11.41

Coordinate differentials

Let (x^i) be coordinates on $U \subseteq M$. For each $x^i : U \to \mathbb{R}$, get coordinate differential $dx^i \in \mathfrak{X}^*(U)$. Then $F : U \to \mathbb{R}$ ~~looks like~~ will look like

$$dF_p = \frac{\partial F}{\partial x^i} dx^i \quad \text{i.e.} \quad dF = \frac{\partial F}{\partial x^i} dx^i$$

$$\text{"} = \frac{dF}{dx} dx \text{"}$$

Pullbacks of differential 1-forms

Let $F : M \to N$ be smooth. We get, for each $p \in M$, a map

$$dF_p : T_pM \to T_{F(p)}N.$$

$$\Longrightarrow dF_p^* : T^*_{F(p)}N \to T_p^*M, \text{ where}$$

dF_p^* is called pullback by F at p.

$$dF_p^*(\omega)(v) = \omega(dF_p(v)) \text{ for } \omega \in T^*_{F(p)}N,$$
$$v \in T_pM.$$

Recall: pushforward of vector fields was fussy, and didn't exist for all smooth maps $F : M \to N$. Naively, might expect