$$dF_p : T_pM \longrightarrow T_{F(p)}\mathbb{R} \cong \mathbb{R}$$

canonical!

$\Rightarrow dF_p \in T_p^*M$

Lecture 10.2

Reading: Finish Ch 11

Practice: 11.32, 11.35, 11.41

Coordinate differentials

Let (x^i) be coordinates on $U \subseteq M$. For each $x^i : U \to \mathbb{R}$, get **coordinate differential** $dx^i \in \mathfrak{X}^*(U)$. Then $F : U \to \mathbb{R}$ ~~looks like~~ will look like

$$dF_p = \frac{\partial F}{\partial x^i} dx^i. \quad \text{i.e.} \quad dF = \frac{\partial F}{\partial x^i} dx^i \;\; \text{"}= \frac{dF}{dx} dx\text{"}.$$

Pullbacks of differential 1-forms

Let $F : M \to N$ be smooth. We get, for each $p \in M$, a map

$$dF_p : T_pM \to T_{F(p)}N.$$

$$\Rightarrow dF_p^* : T^*_{F(p)}N \to T_p^*M, \text{ where}$$

dF_p^* is called **pullback by F at p**.

$$dF_p^*(\omega)(v) = \omega(dF_p(v)) \text{ for } \omega \in T^*_{F(p)}N,$$
$$v \in T_pM.$$

Recall: pushforward of vector fields was fussy, and didn't exist for all smooth maps $F : M \to N$. Naively, might expect something similar for the dual notion: pullbacks of differential 1-forms. However, surprisingly, they ___always exist___!

~~Prop~~ Def If $F: M \to N$ smooth and $\omega \in \mathfrak{X}^*(N)$, define

$$(F^*\omega)_p = dF_p^*(\omega_{F(p)})$$

i.e. $(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v))$ for $v \in T_pM$.

Prop 11.26 If $\omega \in \mathfrak{X}^*(N)$, then $F^*\omega \in \mathfrak{X}^*(M)$.
(i.e. smooth pullback to smooth)

To prove this, first need

Prop 11.25 $F: M \to N$ C^∞, $\omega \in \mathfrak{X}^*(N)$, $u \in$ ~~continuous~~ $\cancel{C(M)}$ $C^\infty(M)$. Then

$$F^*(u\omega) = (u \circ F)\, F^*\omega \quad \text{and} \quad F^* du = d(u \circ F)$$

Proof
$$\begin{aligned} F^*(u\omega)_p &= dF_p^*((u\omega)_{F(p)}) \\ &= dF_p^*(u(F(p))\,\omega_{F(p)}) \\ &= u(F(p))\, dF_p^*(\omega_{F(p)}) \end{aligned}$$

$$= u(F(p))\,(F^*\omega)_p\,.$$

$$= \big((u\circ F)(F^*\omega)\big)_p\,.$$

$$(F^*du)_p(v) = \big(dF_p^*(du_{F(p)})\big)(v)$$

$$= du_{F(p)}(dF_p^* v)$$

$$= dF_p(v)\,u$$

$$= v(u\circ F)$$

$$= d(u\circ F)_p(v).$$

$\square$

<u>Proof of 11.26</u> Let $p\in M$,

Pick coordinates (y^j) on ~~N, let p∈M and~~ neighborhood V of $F(p)$ and let $U = F^{-1}(V)\ni p$.

Write $\omega = \omega_j\,dy^j$. Then

$$F^*\omega = F^*(\omega_j\,dy^j) = (\omega_j\circ F)(F^*dy^j)$$ (☆)

$$= (\omega_j\circ F)\,d(y^j\circ F).$$

Useful: $F^*(\omega_j\,dy^j) = (\omega_j\circ F)\,d(y^j\circ F)$

Since ω_j, F, y^j are smooth, so is $F^*\omega$. $\square$

<u>Example</u> ~~$F:\mathbb{R}^3\to\mathbb{R}^2$~~ $F:\mathbb{S}^1\to\mathbb{R}^3$

$$(p,q)\mapsto(p,q,0)$$

$\omega = dz - y\,dx$. Then $F^*\omega$ computed easily

$$F^*w = d(z\circ F) - (y\circ F)\, d(x\circ F)$$
$$= d(0) - q\, dp$$
$$= q\ - dp.$$

Let $[a,b] \subseteq \mathbb{R}$ and w a 1-form on $\mathbb{R}$. Then the line integral is

$$\int_{[a,b]} w := \int_a^b F(t)\, dt$$

where t is the standard coordinate on $\mathbb{R}$ and $w = F(t)\, dt$ for some $F \in C^\infty(\mathbb{R})$

Line integrals

Let $\gamma : [a,b] \to M$ be a smooth curve and let $w \in \mathfrak{X}^*(M)$. The line integral of w over γ is

$$\int_\gamma w := \int_{[a,b]} \gamma^* w$$

If γ a piecewise smooth curve on $[a,b]$, can define

$$\int_\gamma w := \sum_{i=1}^{k} \int_{[a_{i-1}, a_i]} \gamma^* w,$$

where

$$[a,b] = \bigcup_{i=1}^{k} [a_{i-1}, a_i] \quad \text{and} \quad (a_{i-1}, a_i) \cap (a_{j-1}, a_j) = \emptyset$$

for $i \neq j$.

Def We say $\tilde{\gamma} : [c,d] \to M$ is a reparametrization of $\gamma : [a,b] \to M$ if exists a diffeo $\varphi : \cancel{[a,b]}\ [c,d] \to [a,b]$ such that $\tilde{\gamma} = \gamma \circ \varphi$. If ~~...~~ $\varphi'(t) > 0\ \forall t$, call $\tilde{\gamma}$ as ~~orientek~~ <u>orientation preserving reparametrization</u>. ~~If~~ Otherwise, it is <u>orientation reversing</u>.

<u>**Prop 11.37**</u> Let ~~$\gamma_1 \tilde{\gamma}$~~ $\tilde{\gamma} : [c,d] \to M$ be a reparametrization of $\gamma : [a,b] \to M$ and let $\omega \in \mathfrak{X}^*(M)$.

Then

$$\int_{\tilde{\gamma}} \omega = \begin{cases} \int_{\gamma} \omega & \text{if orientation preserving} \\ -\int_{\gamma} \omega & \text{if orientation reversing.} \end{cases}$$

Proof Write $\tilde{\gamma} = \gamma \circ \varphi$. Then

$$\int_{\tilde{\gamma}} \omega = \int_{[c,d]} \tilde{\gamma}^* \omega = \int_{[c,d]} (\gamma \circ \varphi)^* \omega = \int_{[c,d]} \varphi^* \gamma^* \omega.$$

$\cancel{= \int_{[a,b]} \tilde{\gamma}^* \omega}$ Let $\tilde{\omega} \in \mathfrak{X}([c,d])$. ~~(e.g. $\tilde{\omega} = \gamma^* \omega$)~~.

$\tilde{\omega} = F(t)\,dt = \gamma^* \omega$

$$\cancel{\int_{[c,d]} \varphi^*\tilde{\omega} = \int_{[c,d]} = \int_c^d \varphi^*}$$

$\varphi^*\tilde{\omega} \in \mathfrak{X}^*([a,b])$ defined as

$$\varphi^*\tilde{\omega} = \varphi^*(f(t)\,dt) = f(\varphi(s))\,\varphi'(s)\,ds$$

// 11.25

$$f\circ\varphi(s)\, d(\varphi(s)) \qquad \cancel{f(\varphi(s))}$$

$$\Rightarrow \int_{[c,d]} \varphi^*\gamma^*\omega = \int_c^d f(\varphi(s))\,\varphi'(s)\,ds$$

$$= \int_{\varphi(c)}^{\varphi(d)} f(t)\,dt = \pm \int_a^b f(t)\,dt = \pm \int_{[a,b]} \gamma^*\omega. \quad \square$$

<u>Theorem 11.3</u> (Fundamental Theorem of ~~Calculus~~ Line Integrals) Suppose $F \in C^\infty(M)$ and $\gamma : [a,b] \to M$ (piecewise) smooth curve.

Then
$$\int_\gamma dF = F(\gamma(b)) - F(\gamma(a)).$$

Proof Suffices to assume image of γ in a single chart (x^i) on M. Write

$$\int_\gamma dF = \int_{[a,b]} \gamma^* dF \qquad dF = \frac{\partial F}{\partial x^i} dx^i$$

$\gamma = (\gamma^1, \ldots, \gamma^n)$ in these coordinates.

Before proofing the theorem, a tool:

Prop 11.38 If $\gamma : [a,b] \to M$ smooth (piecewise), then

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)} (\gamma'(t))\, dt$$

Proof Assume γ smooth w/ image in a single chart (x^i) on M. Write $\omega = \omega_i\, dx^i$ and $\gamma = (\gamma^1, \ldots, \gamma^n)$ in these coordinates. Thus

$$(\gamma^* \omega)_t \overset{11.25 \text{ or } (*)}{=} (\omega_i \circ \gamma)(t)\, d(\gamma^i)_t = \omega_i(\gamma(t)) \frac{d\gamma^i}{dt}(t)\, dt$$

$$\cancel{= \omega_i(\gamma(t))\, d\gamma^i(t)}$$

$$= \omega_i(\gamma(t))\, dx^i(\gamma'(t))\, dt$$

$$= \omega_{\gamma(t)} (\gamma'(t))\, dt$$

$$\Rightarrow \int_\gamma \omega = \int_{[a,b]} \gamma^* \omega = \int_a^b \omega_{\gamma(t)} (\gamma'(t))\, dt. \qquad \square$$

Note
$\gamma^i : [a,b] \to \mathbb{R}$
$t \mapsto \gamma^i(t) = x^i(\gamma(t))$;
So $d\gamma^i$ is just differential of this function i.e. $\frac{d\gamma^i}{dt}(t)\, dt$ in coordinates of $\mathbb{R}$

Proof of Thm 11.39

$$\int_\gamma dF = \int_a^b dF_{\gamma(t)}(\gamma'(t))\,dt = \int_a^b (F\circ\gamma)'(t)\,dt$$

$$\underset{\text{FTOC}}{=} F\circ\gamma(b) - F\circ\gamma(a).$$

$\square$