Lecture 2.1

Reading: first half of Ch 2

Practice problems: 1-6, 2.2, 2.3, 2.9, 2.17, 2.16

Homework: 2-1, 2-3, 2-5, 2-6

Start of class: Discuss some loose ends from last time about smooth atlases. Show Lemma 1.35 on overhead projector. "All-knowing atlas" lemma.

Smooth maps

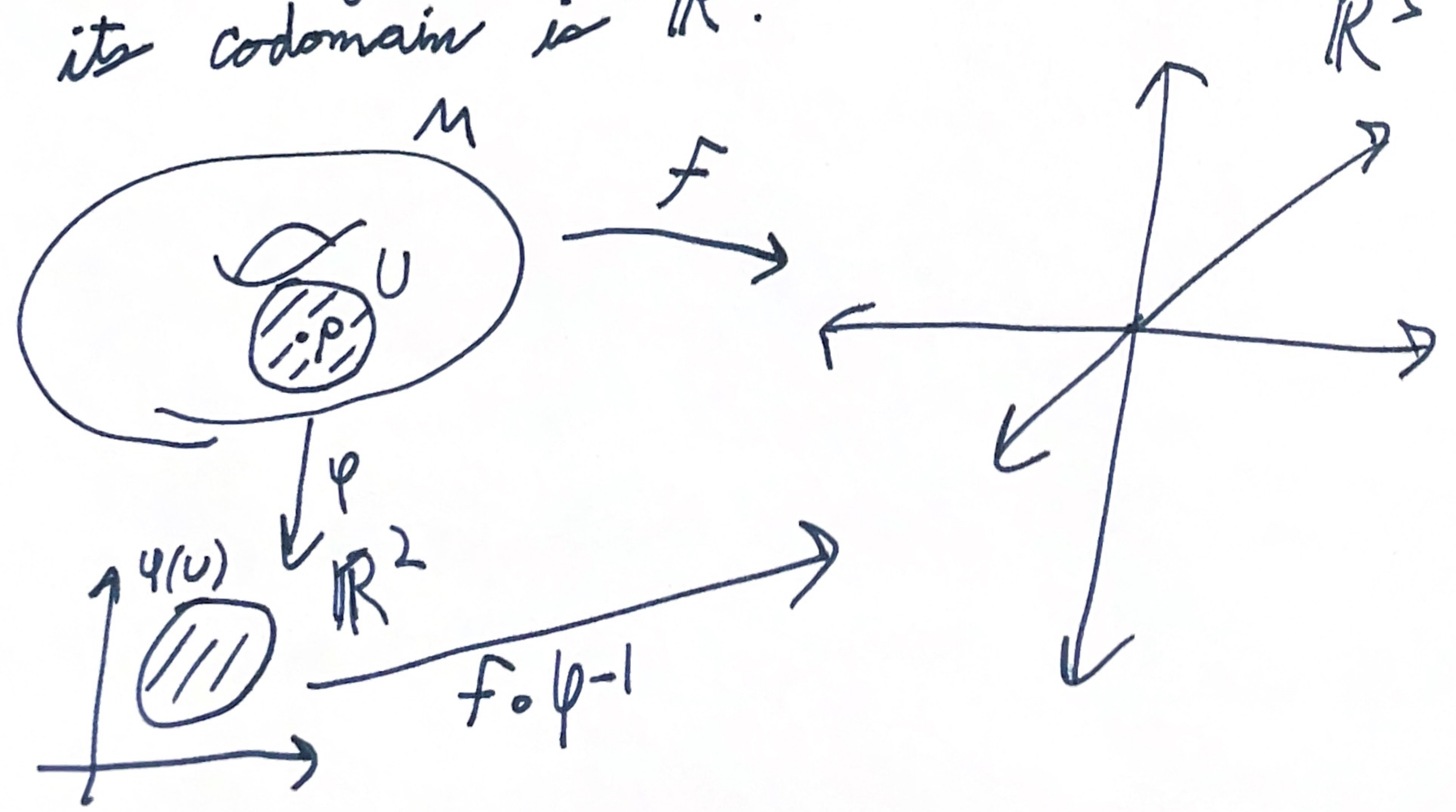
M: C^∞ n-manifold, $f: M \to \mathbb{R}^k$ a function.

Def We say f is smooth if for every $p \in M$, exists a chart (U, φ) of M around p such that

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$$

is smooth.

Note: We know what it means for $f \circ \varphi^{-1}$ to be smooth already, because its domain $\varphi(U) \subseteq \mathbb{R}^n$, and its codomain is $\mathbb{R}^n$.



Def M: C^∞ m-manifold, N: C^∞ n-manifold

$F: M \to N$ ~~function~~ map

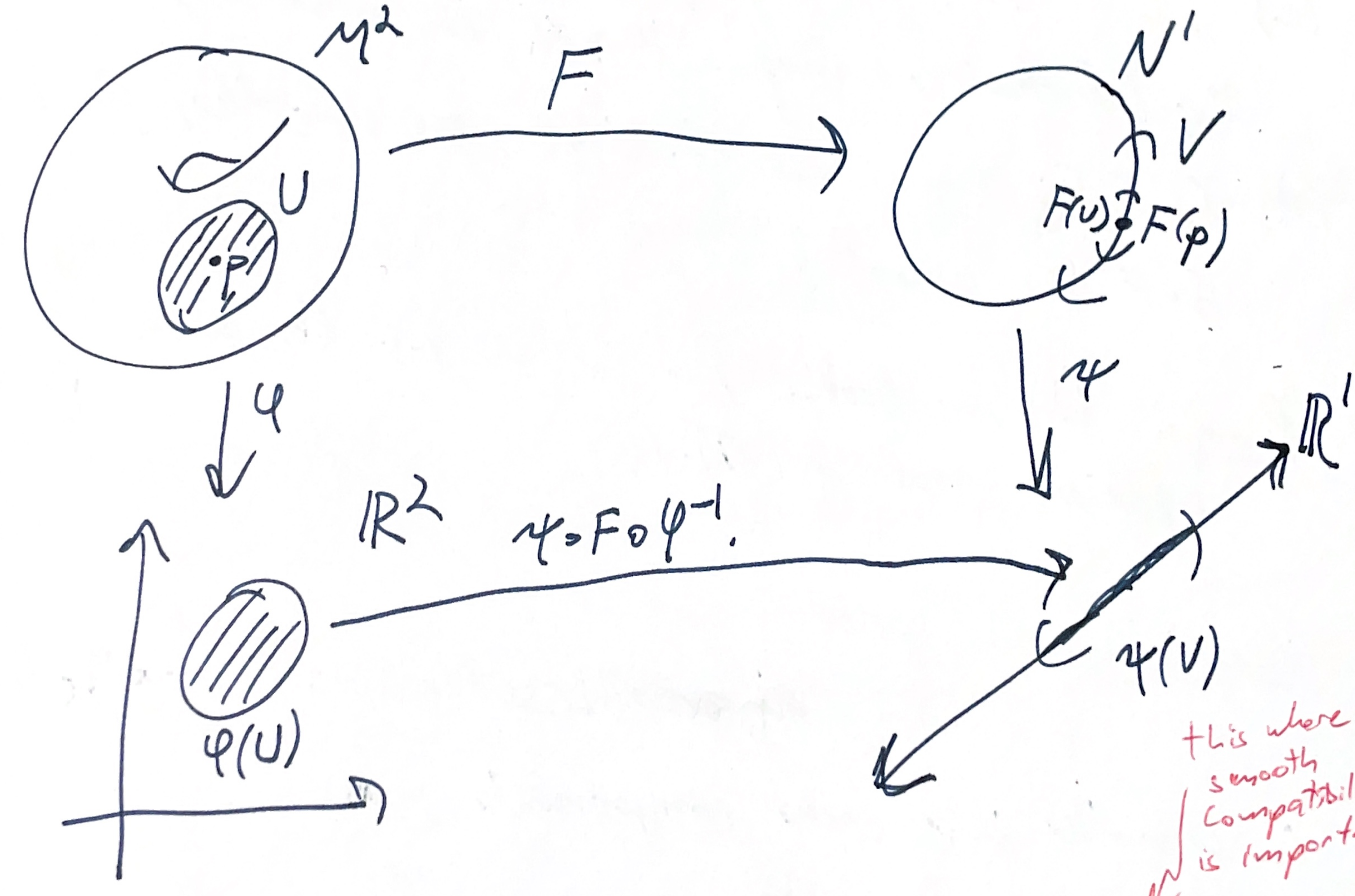
We say F is <u>smooth</u> if for all $p \in M$ there exist charts (U, φ), (V, ψ) in M, N (resp.) such that

(i) $p \in U$

(ii) $F(p) \in V$

(iii) $F(U) \subseteq V$

(iv) $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.



<u>Equivalently</u>: $F: M \to N$ is smooth if for <u>every</u> coordinate chart (V, ψ) on N and every chart (U, φ) on M with $U \subseteq F^{-1}(V)$, the function

$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$ is smooth.

See Props 2.5, 2.6 and Cor. 2.8 for more discussion.

Key intuition: Smoothness is a "local" property.

Examples

Prop. 2.10 Let M, N, P be smooth manifolds.

(a) Every constant map $c: M \to N$ is smooth.

(b) The identity map $id: M \to M$ is smooth.

(c) If $U \subseteq M$ is an open submanifold, then the inclusion map $U \hookrightarrow M$ is smooth.

(d) If $F: M \to N$ and $G: N \to P$ are smooth, then $G \circ F$ is smooth.

Proof: (d) in book. Do (a) or (c). (b) is a special case of ~~(b)~~. (c). $\square$

Prop 2.12 A product of maps

$$F_1 \times \cdots \times F_k : N$$

is smooth iff each component map

$$\pi_i \circ F_i : N \to M_i$$

is smooth.

Ex. 2.13 (d) Inclusion map $S^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+1}$ is smooth.
Does not follow from previous results because S^n is not open in $\mathbb{R}^{n+1}$. Have to look at coordinates of inclusion. Easiest to use orthographic coordinates

$$\varphi_i^{\pm} : U_i^{\pm} \to \mathbb{R}^n$$

where, recall, $U_i^{\pm} = \{ x \in S^n \subseteq \mathbb{R}^{n+1} \mid \operatorname{sign}(x^i) = \pm \}$. In these coordinates, the inclusion map is

$$\varphi_i^+(U_i^+) \xrightarrow{(\varphi_i^+)^{-1}} U_i^+ \overset{i}{\hookrightarrow} \mathbb{R}^{n+1}$$
$$u \longmapsto (u^1, \dots, u^{i-1}, \sqrt{1-|u|^2}, u^i, \dots, u^n)$$

where $(\varphi_i^+)(U_i^+) = \mathbb{B}^n$. Easily checked to be smooth on $\mathbb{B}^n$.

<u>Diffeomorphisms</u>
As hinted in discussion at beginning of class, "equal maximal smooth atlas" is not the "correct" notion of equivalence of smooth manifolds. Here is correct notion:
<u>Def</u> A smooth map $F: M \to N$ is a <u>diffeomorph</u> if F^{-1} exists and is smooth.

Note: $F: \mathbb{R} \to \mathbb{R}$ is a smooth homeomorphism
$x \mapsto x^3$

but not a diffeomorphism.

Ex $F: \mathbb{B}^n(r) \to \mathbb{B}^n$ is a diffeomorphism.
$x \mapsto \frac{x}{r}$

Ex $F: \mathbb{B}^n \to \mathbb{R}^n$ is a diffeomorphism.
$x \mapsto \frac{x}{\sqrt{1-|x|^2}}$

Rmk Could thus modify definition of smooth manifold to require charts be onto functions $\varphi: U \to \mathbb{R}^n$ without getting anything less restrictive.

Prop 2.15
(a) Diffeomorphisms compose to diffeomorphisms.
(b) Products of diffeomorphisms are diffeomorphisms
(c) Diffeomorphism $\Rightarrow$ homeomorphism
(in particular, open)
(d) Restriction of a diffeomorphism to an open submanifold yields a diffeomorphism onto the image.
(e) "Diffeomorphic" is an equivalence relation.

Theorem If $M^m \cong N^n$, then $m = n$.

Proof Ultimately boils down to the fact that if $L : \mathbb{R}^m \to \mathbb{R}^n$ is an isomorphism of vector spaces, then $m = n$. How? (Derivative!) $\square$

Lecture 2.2

Reading: second half of Ch. 2

Practice problems: 2-27

HW: 2-14

Today we'll prove existence of partitions of unity. These are an important tool for establishing existence of various "global" objects from "local" assumptions. (Moreover, the existence is a big reason we assume that manifolds are paracompact/second-countable.) As a warm-up to this idea, let me recall a fact from point-set topology that we are about to extend to the smooth setting.