Theorem If $M^m \cong N^n$, then $m=n$.

Proof Ultimately boils down to the fact that if $L: \mathbb{R}^m \to \mathbb{R}^n$ is an isomorphism of vector spaces, then $m=n$. How? (Derivative!) $\square$

Lecture 2.2

Reading: second half of Ch. 2

Practice problems: 2-27

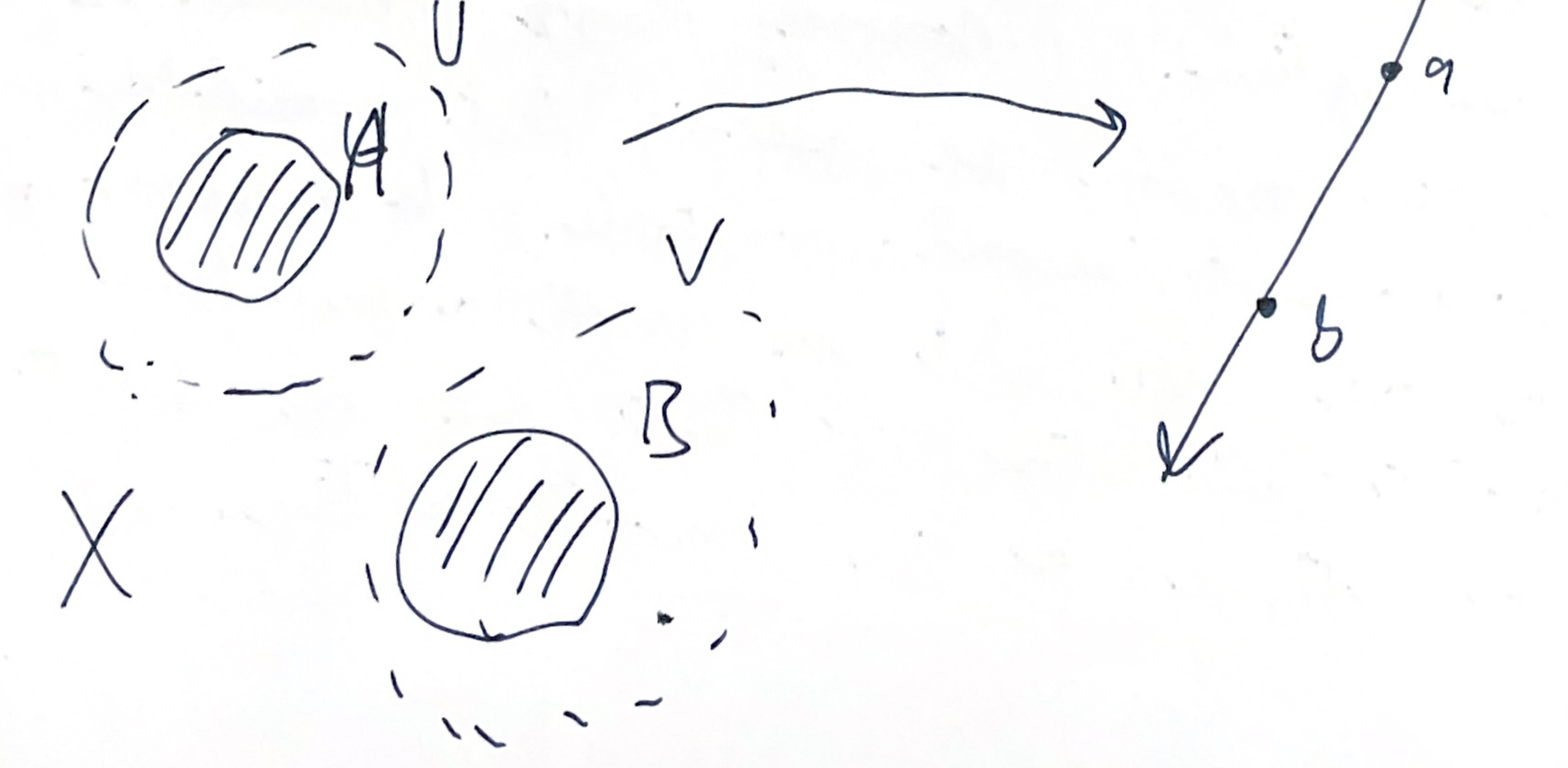
HW: 2-14

Today we'll prove existence of partitions of unity. These are an important tool for establishing existence of various "global" objects from "local" assumptions. (Moreover, the existence is a big reason we assume that manifolds are paracompact / second-countable.) As a warm-up to this idea, let me recall a fact from point-set topology that we are about to extend to the smooth setting.

Def A topological space X is *normal* (or T_4) if for every pair of disjoint closed sets $A, B \subset X$, there existence disjoint open sets $U, V \subset X$ with $A \subseteq U$ and $B \subseteq V$.

Def A pair of disjoint subsets $A, B \subset X$ is *separated* by a function $f : X \to \mathbb{R}$ if exist $a, b \in \mathbb{R}$ with $a \neq b$ and $f(A) = a$, $f(B) = b$.

Thm (Urysohn's lemma)
A topological space X is normal iff every pair of disjoint closed subsets $A, B \subset X$ can be separated by a continuous function $f : X \to \mathbb{R}$.



On Prob 2-14 of HW2, you'll prove a version of this for smooth manifolds and smooth separating functions.

Partitions of Unity

Def Let M be any topological space and let $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be any (indexed) open cover. A partition of unity subordinate to $\mathcal{X}$ is any family of continuous functions

$$\psi_a : M \to \mathbb{R}$$

such that:

(i) $0 \leq \psi_a(x) \leq 1$ for all $\alpha \in A$ and $x \in M$

(ii) $\operatorname{supp} \psi_a := \overline{\{x \in M \mid \psi_a(x) \neq 0\}} \subseteq X_a$

(iii) $\forall x \in M$, $\exists U \subseteq M$ open such that there exist only finitely many $a \in A$ such that $U \cap \operatorname{supp} \psi_a \neq \phi$.

(iv) $\sum_{a \in A} \psi_a(x) = 1 \quad \forall x \in M$.

Def If M is smooth manifold and all ψ_a are smooth, call this a smooth partition of unity (subordinate to $\mathcal{X}$).

Thm 2.23 Suppose M is a smooth manifold, and $\mathcal{X}$ an open cover. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

Before proving this, need several preliminary ideas.

Lem 2.20 The function $f:\mathbb{R}\to\mathbb{R}$

$$f(t) = \begin{cases} e^{-1/t}, & t>0 \\ 0, & t\leq 0 \end{cases}$$

is smooth. (Not analytic!)

Proof sketch: Clearly continuous everywhere and smooth for $t\neq 0$. To show smooth at $t=0$, use L'Hôpital's rule to show all derivatives (i.e. of all order) at $t=0$ from the right vanish. $\square$

Lem 2.21 For any $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, exists an $h:\mathbb{R}\to\mathbb{R}$ such that

- h smooth
- $h(t) = 1$ for $t \leq r_1$
- $h(t) = 0$ for $t \geq r_2$
- $0 < h(t) < 1$ for $r_1 < t < r_2$
- h is monotone

"smooth cutoff function"

Proof: Let

$$h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$$

where f is from Lem 2.20. $\square$

Lem 2.22 For any $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 < r_2$, exists an $H : \mathbb{R}^n \to \mathbb{R}$ such that

- H smooth
- $H(x) = 1$ for $|x| \leq r_1$
- $H(x) = 0$ for $|x| \geq r_2$
- $0 < H(x) < 1$ for $r_1 < |x| < r_2$

"smooth bump function"

Proof Let $H(x) = h(|x|)$, where h from Lem. 2.21. Clearly smooth away from $x = 0$. at $x = 0$, smooth because $r_1 > 0$. $\square$

Def A topological space is paracompact if every open cover admits a open refinement that is locally finite, meaning every point has a neighborhood intersecting at most finitely many sets in the cover.

Thm 1.15 Topological manifolds are paracompact. In fact, if B is any basis of the topology, then the refinement can happen in B.

I won't prove this, but it requires second-countability.

Prop 1.19 Every smooth manifold has a basis consisting of regular coordinate balls, i.e. open sets U in M that admit coordinate charts that extend to a smooth diffeomorphism $\varphi : U \xrightarrow{\cong} B^n$

$\bar{\varphi} : \bar{U} \xrightarrow{\cong} \bar{B^n}$.

Equivalently: exist $U' \supseteq U$ and a chart $\varphi' : U' \to B^n(1+\varepsilon)$ such that $\varphi'|U = \varphi$.

Proof: Exercise 1.19. $\square$

Proof of Theorem 2.23: Theorem 1.15 and Prop 1.19 imply – after some fussing with indices – that we might as well assume $\mathcal{X} = \{X_\alpha\}_{\alpha \in A} = \{B_i\}_{i \in I}$ is a countable, locally finite cover by regular coordinate balls. Note well (important for point (iii) in definition of partition of unity): closures $\bar{B_i}$ also form a locally finite cover.

For each $i \in I$, ~~defining~~ let $f_i : M \to \mathbb{R}$ be a smooth, non-negative function on M that is non-vanishing on B_i. We can build such f_i using Lem 2.22 and the fact B_i is regular.

Let $f : M \to \mathbb{R}$ be defined by $f(x) = \sum_{i \in I} f_i(x)$. This is smooth around any $x \in M$ because x has a neighborhood where only finitely many f_i vanish

Finally, let $g_i(x) = \frac{f_i(x)}{F(x)}$. Then can check that $\{g_i\}$ is a smooth partition of unity subordinate to $\{B_i\}_{i \in I}$. $\square$

Many applications! See Chapter 2. E.g.:

<u>Prop 2.25</u> Let M be a smooth manifold, $A \subseteq M$ closed set, $U \supseteq A$ an open neighborhood of A. Then there exists a function $f : M \to \mathbb{R}$ such that

- $f(a) = 1 \quad \forall a \in A$
- $\operatorname{supp} f \subseteq U$
- $0 \leq f(x) \leq 1 \quad \forall x \in M$
- f is smooth.

Call such an f a <u>smooth bump function for A supported in U.</u>

<u>Proof</u>: Let $\{U_0 = U,\ U_1 = M - A$. Take a partition of unity $\{\psi_0, \psi_1\}$ subordinate to $\{U_0, U_1\}$. Let $f = \psi_0$. $\square$