Lecture 2.3.

Reading: First half of Ch. 3, pp. 50 - 64

Practice: 3.5, 3.7

HW: 3-1, 3-4, 3-5

Begin class with proof of existence of partitions of unity.

~~Dots~~ Points, vectors and arrows in $\mathbb{R}^n$:

Often conflate points in $\mathbb{R}^n$ with vectors in $\mathbb{R}^n$. We're going to stop doing this because we want to think of arrows differently - as tangent vectors to points!

Def The geometric tangent space to a point $p \in \mathbb{R}^n$ is the set

$$\mathbb{R}^n_p := \{(p, v) \mid v \in \mathbb{R}^n\},$$

which we turn into a vector space by declaring $(p, v) + (p, w) = (a, v+w)$.

Note: We will never add (p, v) and (q, w) if $p \neq q$. (Even though we could!)

The <u>geometric tangent bundle</u> for $\mathbb{R}^n$ is the set

$$T\mathbb{R}^n = \bigsqcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p = \bigsqcup_{p \in \mathbb{R}^n} \{p\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

considered as a smooth manifold together with a natural map $\pi: T\mathbb{R}^n \to \mathbb{R}^n$.
$$(p, v) \mapsto p$$

Our goal now is to define these types of structures on arbitrary smooth manifolds.

<u>Motivation</u>: tangent vectors are things that allow us to "do calculus" in $\mathbb{R}^n$ — namely, by taking directional derivatives of smooth function $F: \mathbb{R}^n \to \mathbb{R}$.

<u>Complication</u>: How to define "directions" in an arbitrary M?

<u>One idea (more geometric)</u>: Look at smooth functions (called smooth curves in M) and $\gamma, \gamma': \mathbb{R} \to M$ declare $[\gamma] = [\gamma']$ if $\gamma(0) = \gamma'(0)$ and for every smooth function $F: M \to \mathbb{R}$, $(F \circ \gamma)'(0) = (F \circ \gamma')'(0)$. You'll explore this on HW3.

Another idea (more algebraic/analytic): Recognize that if tangent vectors are expected to be useful because they allow us to take directional derivatives, then we might just try to define directional derivatives directly and identify those with our tangent vectors by definition

Def If M a smooth manifold, then

$$C^\infty(M) := \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}.$$

Note $C^\infty(M)$ is a vector space; finite dimensional iff M is zero dimensional and compact (i.e., iff M is a finite set of points).

Def A linear map $v : C^\infty(M) \to \mathbb{R}$ is a derivation at p if it satisfies

$$v(fg) = f(p)\,vg + g(p)\,vf \qquad \text{"product rule"}$$

for all $f, g \in C^\infty(M)$.

Def The tangent space T_pM to M at p is defined to be the set of all derivations at p.

Why is this sensible?!

Let's give some first justifications.

Ex: T_pM is a vector space.

Lem 3.1 If $w \in T_p\mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$, then

(a) $wf = 0$ for constant f

(b) $w(fg) = 0$ whenever $f(p) = g(p) = 0$.

Proof For (b) just use the assumptions and plug in to the fact that w is a derivation (hence, can use product rule).

For (a), can reduce to $f \equiv 1$. Then apply product rule to ff. □

End on 8/30

Def For $(a, v) \in \mathbb{R}^n_p$, define the directional derivative

$$D_v|_p : C^\infty(\mathbb{R}) \to \mathbb{R}$$

by

$$D_v|_p f = D_v f(p) = \frac{d}{dt}\Big|_{t=0} f(p + tv)$$

$$= \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}.$$

Prop 3.2 Let $p \in \mathbb{R}^n$.

(a) For all $(p,v) \in \mathbb{R}^n_p$, $D_v|_p$ is a derivation

(b) The map

$$\begin{aligned} \mathbb{R}^n_p &\longrightarrow T_p\mathbb{R}^n \\ (p,v) &\longmapsto D_v|_p \end{aligned}$$

is an isomorphism. In particular, $T_p\mathbb{R}^n$ is n-dimensional!

Proof (a) is just checking definitions

(b) uses Taylor's theorem. See book. $\square$

Derivatives of smooth maps

Def Let M, N be smooth manifolds and $F: M \to N$ be smooth. The differential (or (total) derivative) at p is the function

$$dF_p : T_pM \to T_{F(p)}N$$

that takes a derivation $v \in T_pM$ to the derivation on N at $F(p)$ defined by

$$dF_p(v)(f) := v(f\circ F).$$

Ex Check that $dF_p(v)$ is a derivation of N at p.

Prop 3.6 Let $F: M \to N$, $G: N \to P$ be smooth and $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ is linear.

(b) $d(G\circ F)_p = dG_{F(p)} \circ dF_p$ (chain rule)

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}$

(d) If F is a diffeomorphism, then dF_p is an isomorphism, with $dF_p^{-1} = \left(dF_{F^{-1}(p)}\right)^{-1}$.

Moral issue: as we've defined things, derivations of M at p require understanding smooth functions that are globally defined on M. However we expect tangent spaces, intuitively, should depend on local structure in a neighborhood of a point.

Using bump functions, can prove:

Prop 3.9 Let $U \subseteq M$ be open and let $i: U \hookrightarrow M$ be inclusion map. Then for all $p \in U$, $di_p : T_p U \to T_p M$ is an isomorphism.

In particular, by picking U to be a coordinate chart, can compose with the diffeomorphism $\varphi: U \to \varphi(U) \subseteq \mathbb{R}^n$ to determine:

Prop 3.10 If M is a smooth n-manifold then all of its tangent spaces are n-dimensional. $\square$