Lecture 3.2

Reading: Second half of Ch. 3

Practice: Ex 3.19,

HW: Prob 3-8

Announcements:

- HW3 was edited after it went up (3-8, not 3-7)
- Quiz 1 is on Friday!

Derivatives in coordinates

M: C^∞ n-manifold, (U,φ): chart on M

Note that for $p \in U$, $d\varphi_p : T_pM \xrightarrow{\cong} T_{\varphi(p)}\mathbb{R}^n$.

Since $\left\{ \frac{\partial}{\partial x_1}\Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x_n}\Big|_{\varphi(p)} \right\}$ a basis of $T_{\varphi(p)}\mathbb{R}^n$,

$$\underbrace{\left\{ (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x_1}\Big|_{\varphi(p)}\right), \dots, (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x_n}\Big|_{\varphi(p)}\right) \right\}}_{\text{"standard basis" wrt } (U,\varphi)} \text{ a basis of } T_pM.$$

When coordinates (U,φ) understood, often, lazily, write $\frac{\partial}{\partial x_i}$ instead of $(d\varphi_p)^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right)$. Given $F \in C^\infty(U)$, have

$$\text{"}\frac{\partial}{\partial x_i}\Big|_p F\text{"} = (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) F = \frac{\partial}{\partial x_i}\Big|_{\varphi(p)} (F \circ \varphi^{-1})$$

More generally, if $F : M^m \to N^n$ smooth, (U,φ),

(V, ψ) charts on M, N, resp.) then, in coordinates, have

$$dF_p \left(\frac{\partial}{\partial x^i}\Big|_p \right) f = \frac{\partial}{\partial x^i}\Big|_p (f \circ F) \quad \text{(careful!)}$$

$$= \sum_j \frac{\partial f}{\partial y^j} (F(p)) \frac{\partial F^j}{\partial x^i} (p) \quad \text{(chain rule)}$$

$$= \frac{\partial f}{\partial y^j} (F(p)) \frac{\partial F^j}{\partial x^i} (p) \quad \text{(Einstein notation)}$$

$$= \left(\frac{\partial F^j}{\partial x^i} (p) \frac{\partial}{\partial y^j}\Big|_{F(p)} \right) f$$

where $y^1, \dots, y^m$ coordinates on N from (V, ψ) and $f \in C^\infty(N)$. In other words, in standard bases induced from φ, ψ, have

$$dF_p \text{ "=" } \begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

where you must get comfortable with

the abused notation, i.e. really have

$$"\frac{\partial F^j}{\partial x^i}(p)" = \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} (\psi \circ F \circ \varphi^{-1})^j.$$

Velocity vectors of curves

A curve in M is a smooth map

$$\gamma : J \to M$$

where $J \subseteq \mathbb{R}$ is an (open) interval. Standard Coordinate on J is usual called t, not x. The velocity vector of γ at t_0 is defined to be

$$\gamma'(t_0) := d\gamma\left(\frac{d}{dt}\bigg|_{t_0}\right) \in T_{\gamma(t_0)}M.$$

Note that $\gamma'(t_0) F = \frac{d}{dt}\bigg|_{t_0} (F \circ \gamma) = (F \circ \gamma)'(t_0)$

for $F \in C^\infty(M)$.

Prop 3.23 If $v \in T_pM$ is the velocity vector of some curve $\gamma : J \to M$.

Proof: Let (U, φ) be a coordinate ball centered at p. Suppose $\varphi(U) = \mathbb{B}^n_\varepsilon$, and let $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to \mathbb{B}^n_\varepsilon$ be defined by $\tilde{\gamma}(t) = t\tilde{v}$, where

$\tilde{v} = (d\varphi)_p v$. Then $\gamma = \varphi^{-1} \circ \tilde{\gamma}$ is a smooth curve in M with $\gamma'(0) = v$. $\square$

Prop 3.24 If $F: M \to N$ smooth and $\gamma: J \to M$ curve, then $F \circ \gamma$ a curve with

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

$\square$

Proof Chain rule.

Corollary 3.25 If $F: M \to N$ smooth and $v \in T_p M$, then

$$dF_p\, v = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma: J \to M$ with $\gamma'(0) = v$.