Lecture 3.3

Reading: finishing Ch. 3. Start on Ch. 4 over weekend

Practice: n/a. Ex. 3.27 if interested in categories

Tangent bundles and <u>the</u> derivative

Def The tangent bundle of M is the set

$$TM := \bigsqcup_{p \in M} T_p M$$

together with the natural projection map $\pi : TM \to M$ defined by $\pi(v) = p$ if $v \in T_p M$.

Proposition TM has a natural smooth structure making it into a smooth manifold of dimension $2n$. The natural projection is smooth w.r.t. this smooth structure.

Proof: Given a smooth coordinate chart (U, φ) on M, we show how to "promote" it to a chart $\tilde{\varphi}$ on $\pi^{-1}(U) = \bigsqcup_{p \in U} T_p M$. Simply:

$$\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$$
$$v \mapsto (\varphi(\pi(v)), v^1, v^2, \dots, v^n)$$

where $v = \sum_{j=1}^{n} v^j \frac{\partial}{\partial x^j}\Big|_{\pi(v)}$.

If (U, φ) and (V, ψ) are two smoothly compatible charts on M, we want to show that $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ are smoothly compatible charts on TM. We must analyze transition maps:

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$$
$$(x, v) \mapsto \left(\psi \circ \varphi^{-1}(x), \, d(\psi \circ \varphi^{-1})_x(v)\right)$$

The component functions of $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ are

$\psi \circ \varphi^{-1} \circ p$ (where $p : \varphi(U \cap V) \times \mathbb{R}^n \to \varphi(U \cap V)$, $(x, v) \mapsto x$)

and

$$\left[d(\psi \circ \varphi^{-1})_x(v)\right]^i = \sum_{j=1}^{n} \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial x^j}(x)\, v^j \in \mathbb{R}$$

$\psi \circ \varphi^{-1}$ is smooth by assumption and p is easily checked to be smooth; hence $\psi \circ \varphi^{-1} \circ p$ is smooth.

The function $\varphi(U \cap V) \times \mathbb{R}^n \to \mathbb{R}$

$$(x, v) \mapsto \sum_j \frac{\partial (\psi \circ \varphi^{-1})^i}{\partial x^j}(x)\, v^j$$

is easily verified to be smooth.

We conclude that the transition function $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth. Clearly, if $\{U\}$ is an open cover of M, then $\{\pi^{-1}(U)\}$ is an open cover of TM. Thus, any smooth atlas on M can get promoted to a smooth atlas on TM. In these "promoted" or "natural" coordinates of TM, the map $\pi : TM \to M$ just looks like $(x, v) \mapsto x$, which is smooth. Hausdorff? Second-countable? $\square$

<u>Rmks</u>

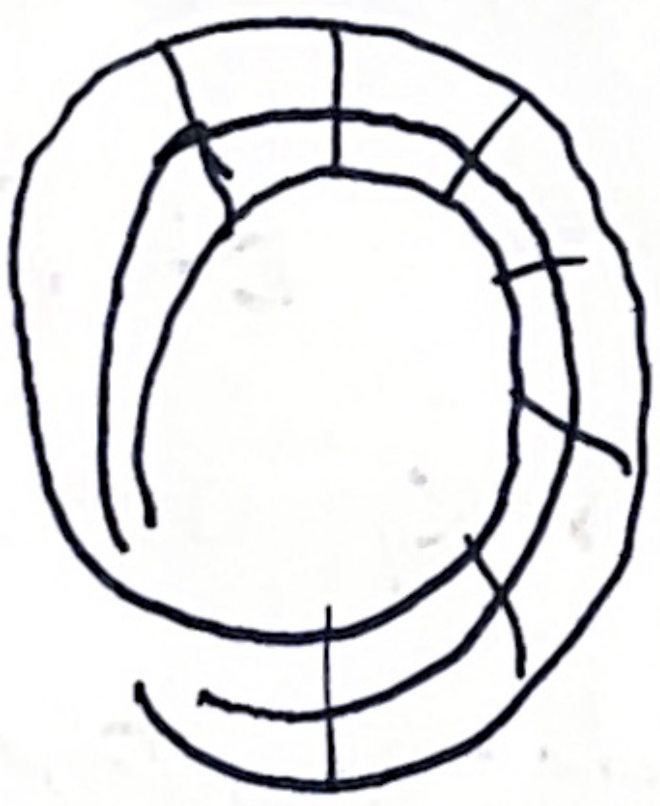
- Intuitively, if $p \in M$, and $F \in C^\infty(M)$, then depends smoothly on p.

$$\left.\frac{\partial}{\partial x^i}\right|_p F$$

- More precisely, if M is a C^k manifold, then TM is a C^{k-1} manifold.
- For "most" manifolds, $TM \not\cong M \times \mathbb{R}^n$!

- TM is an example of a vector bundle. There are other "canonical" examples of vector bundles associated to smooth manifolds. Most important: $\Lambda^k T^*M$.

"Non-canonical" example: Möbius strip is a $\mathbb{R}$ bundle over S^1. It is <u>not</u> diffeomorphic to $S^1 \times \mathbb{R}$. (However, $TS^1 \cong S^1 \times \mathbb{R}$.)



<u>Later</u>: $TS^2 \not\cong S^2 \times \mathbb{R}^2$. In fact, follows from "hairy ball theorem".

<u>Def</u> If $F: M \to N$ is smooth, the <u>derivative</u> is

$$dF: TM \to TN$$

$$(p,v) \mapsto (F(p), dF_p(v)).$$

Prop 3.21 If $F: M \to N$ smooth, then $dF: TM \to TN$ is smooth. Moreover, following diagrams commute

$$\begin{tikzcd} TM \arrow[r, "dF"] \arrow[d, "\pi"'] & TN \arrow[d, "\pi"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

$$\begin{tikzcd} T_pM \arrow[r, "dF_p"] \arrow[d, "i"] & T_{F(p)}N \arrow[d, "i"] \\ TM \arrow[r, "dF"'] & TN \end{tikzcd}$$

Why is $i: T_pM \to TM$ smooth?

Proof: In natural coordinates,

$$dF(x^1, \dots, x^n, v^1, \dots, v^n) = \left(F^1(x), F^2(x), \dots, F^n(x), \frac{\partial F^1}{\partial x^i}(x) v^i, \dots \frac{\partial F^n}{\partial x^i}(x) v^i\right)$$

which is clearly smooth. (because F is).

The diagrams are an exercise. $\square$

Corollary 3.22 Let $F: M \to N$, $G: N \to P$ be smooth.

(a) $d(G \circ F) = dG \circ dF$

(b) $d(Id_M) = Id_{TM}$

(c) If F is a diffeomorphism, then $dF: TM \to TN$ is a diffeomorphism and $d(F^{-1}) = (dF)^{-1}$.