Corollary 3.22 Let $F: M \to N$, $G: N \to P$ be smooth.

(a) $d(G \circ F) = dG \circ dF$

(b) $d(Id_M) = Id_{TM}$

(c) If F is a diffeomorphism, then $dF: TM \to TN$ is a diffeomorphism and $d(F^{-1}) = (dF)^{-1}$.

Lecture 4.1

Reading: Ch. 4

Practice: Ex. 4.3, 4.4, 4.7, 4.10

HW: 4-4

Maps of Constant Rank

Def $F: M^m \to N^n$ has constant rank r if $\operatorname{rank} dF_p = r$ for all $p \in M$. F has full rank if it has constant rank

$$r = \min\{m, n\}.$$

A full rank F is a submersion if its rank is n; it is an immersion if its rank is m.

Equivalently: $F: M \to N$ is a <u>submersion</u> if dF_p is surjective for all $p \in M$. It is an <u>immersion</u> if dF_p is injective for all $p \in M$.

<u>Prop 4.1</u> Suppose $F: M \to N$ smooth and $p \in M$. If dF_p surjective, then $F|_U$ is a submersion for some open $U \ni p$. If dF_p ~~at~~ is injective, then $F|_U$ is an immersion for some $U \ni p$.

<u>Proof</u> Picking coordinates (U, φ), (V, ψ) gives a map

$$dF_{\_}: U \to M(m \times n, \mathbb{R})$$
$$q \mapsto dF_q.$$

Since $dF_{\_}$ is continuous (in fact, smooth), and $\{A \in M(m \times n, \mathbb{R}) \mid A \text{ has full rank}\}$ is open, we see that dF_q has full rank in some neighborhood of p. $\square$

<u>Most important examples</u>

1.
$$\pi: \mathbb{R}^{n+k} \to \mathbb{R}^n$$
$$(x^1, \dots, x^{n+k}) \mapsto (x^1, \dots, x^n)$$

is a submersion.

2.
$$i: \mathbb{R}^n \to \mathbb{R}^{n+k}$$
$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0)$$ is an immersion.

The "rank theorem" says that all full rank smooth maps admit local coordinates that look like these.

<u>More examples</u>

- $\gamma : J \to M$ a smooth curve is an immersion if and only if $\gamma'(t) \neq \vec{0}$ for all $t \in J$.

- $\pi : TM \to M$ (natural projection) is a submersion.

<u>Def</u> $F : M \to N$ is a local <u>diffeomorphism</u> if every $p \in M$ has an open neighborhood U such that $F(U)$ is open and $F|_U : U \to F(U)$ is a diffeomorphism.

<u>Thm 4.5 (Inverse function theorem)</u>
Suppose $F : M \to N$ smooth and $p \in M$ such that dF_p is invertible. Then there exist open neighborhoods U of p and V of $F(p)$ such that $F|_U : U \to V$ is a diffeomorphism.

<u>Proof</u> Use coordinates and apply inverse function theorem in $\mathbb{R}^n$. $\square$

Prop 4.8 Suppose $F: M \to N$ smooth.

(a) F is a local diffeomorphisms iff it is both a submersion and an immersion.

(b) If $\dim M = \dim N$ and F has full rank, then F ~~it~~ is a local diffeomorphism.

Proof ... $\square$

See Prop. 4.6 for a bunch of easy/useful properties of local diffeomorphisms.

Rank theorem

More-or-less completely describes the local structure of smooth maps of constant rank.

Theorem Suppose $F: M^m \to N^n$ smooth of constant rank $r \leq \min\{m, n\}$. For each $p \in M$, there exist coordinates (U, φ) centered at p and (V, ψ) centered at $F(p)$ such that $F(U) \subseteq V$ in which the coordinate representation of F is

$$F(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, \underbrace{0, \dots, 0}_{n-r}).$$

In particular, if F a submersion, have

$$F(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n)$$

and if an immersion $F(x^1, \dots, x^m) = (x^1, \dots, x^m, \overbrace{0, \dots, 0}^{n-m})$

We prove this Wednesday. For now:

Corollary Suppose M connected and $F: M \to N$ smooth. TFAE:

(a) for each $p \in M$, there exist coordinates so that F is linear.

(b) F has constant rank.

Proof (a) $\Rightarrow$ (b): Use connected and the ~~f~~ assumption

(b) $\Rightarrow$ (a) Use rank theorem. $\square$

Lecture 4.2

Reading: Finish Ch. 4. Don't worry too much about section on covering maps.

Practice: 4.16, 4.24, 4.32

HW: 4-4, 4-7, 4-8.

Restate Rank Theorem.

Proof Three steps:

1. Preliminaries: reduce to a problem in $\mathbb{R}^m$ and $\mathbb{R}^n$ with some obvious conveniences.

2. Use IFT to find even nicer coordinates ~~where~~ on the domain where the ~~function~~ map looks like coordinate projection composed with "inclusion into a graph."