We prove this Wednesday. For now:

Corollary Suppose $M$ connected and $F: M \to N$ smooth. TFAE:

(a) for each $p \in M$, there exist coordinates so that $F$ is linear.

(b) $F$ has constant rank.

Proof (a) $\Rightarrow$ (b): Use connected and the ~~f~~ assumption

(b) $\Rightarrow$ (a) Use rank theorem. $\square$

## Lecture 4.2

Reading: Finish Ch. 4. Don't worry too much about section on covering maps.

Practice: 4.16, 4.24, 4.32

HW: 4-4, 4-7, 4-8.

Restate Rank Theorem.

Proof Three steps:

1. Preliminaries: reduce to a problem in $\mathbb{R}^m$ and $\mathbb{R}^n$ with some obvious conveniences.

2. Use IFT to find even nicer coordinates ~~where~~ on the domain where the ~~function~~ map looks like coordinate projection composed with "inclusion into a graph"

$$(x^1,\dots,x^r,x^{r+1},\dots,x^m) \mapsto (x^1,\dots,x^r, S(x^1,\dots,x^r))$$

for a smooth map $S: \mathbb{R}^r \to \mathbb{R}^{n-r}$.

3. Find nicer coordinates on the codomain that allow us to take $S \equiv 0$.

The above is just a rough sketch. I'll explain steps 1 and 2 in detail.

<u>Step 1</u> Suffices to assume $M = U \subseteq \mathbb{R}^m$, $N = V \subseteq \mathbb{R}^n$, $p = 0 \in U$ and $F(p) = 0 \in V$. Since $dF_p$ has rank $r$, can arrange so the Jacobian matrix $DF_0$ has its upper left $r \times r$ block invertible. For convenience, we'll write points in $\mathbb{R}^m$ as $(x^1,\dots,x^r,y^1,\dots,y^{m-r})$ and points in $\mathbb{R}^n$ as $(v^1,\dots,v^r,w^1,\dots,w^{n-r})$, and

$$F(x,y) = (Q(x,y), R(x,y))$$

for $Q: U \to \mathbb{R}^r$ and $R: U \to \mathbb{R}^{n-r}$.

<u>Step 2</u> Define "pre" coordinate chart via

$$\varphi: U \to \mathbb{R}^m$$
$$(x,y) \mapsto (Q(x,y), y).$$

Note $D\varphi_{(0,0)} = D\varphi_p = \left(\begin{array}{c|c} \frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q^i}{\partial y^j}(0,0) \\ \hline 0 & \delta^i_j \end{array}\right)$ is invertible, hence

IFT $\Rightarrow$ exist $U_0 \subseteq U$ and $\tilde{U}_0 \subseteq \mathbb{R}^m$ such that

$\varphi|_{U_0} : U_0 \to \tilde{U}_0$ is a diffeomorphism.

Can show without too much effort that

$$(\varphi|_{U_0})^{-1}(x,y) = (A(x,y), y)$$

for $A: \tilde{U}_0 \to \mathbb{R}^r$ smooth.

$\varphi \circ \varphi^{-1} = Id \Rightarrow Q(A(x,y),y) = x$

$\Rightarrow F \circ \varphi^{-1}(x,y) = (x, \tilde{R}(x,y))$ for $\tilde{R} : \tilde{U}_0 \to \mathbb{R}^{n-r}$ smooth $(\tilde{R}(x,y) = R(A(x,y),y))$.

$$D(F \circ \varphi^{-1})(x,y) = \left( \begin{array}{c|c} I_r & 0 \\ \hline \frac{\partial \tilde{R}^i}{\partial x^j}(x,y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x,y) \end{array} \right)$$

which has rank $r$ and first $r$ columns linearly independent. Thus

$$\frac{\partial \tilde{R}^i}{\partial y^j}(x,y) = 0$$

for all $i = 1, \dots, n-r$ and $j = 1, \dots, n-r$. Thus

$$\tilde{R}(x,y) = \tilde{R}(x,0) \qquad \forall (x,y) \in \tilde{U}_0,$$

so define $S(x) = \tilde{R}(x,0)$. Then

$$F \circ \varphi^{-1}(x,y) = (x, S(x)). \qquad (*)$$

Step 3 We can restrict $\tilde{U}_0$ to be a product

$$\tilde{U}_0 = \mathbb{B}^r_\varepsilon \times \mathbb{B}^{m-r}_\varepsilon, \quad \text{for some small } \varepsilon > 0.$$

Let $V_0 = \{(v,w) \in V \mid (v,0) \in \tilde{U}_0\} \subseteq V$. Note $V_0$ open and $F(p) = (0,0) \in V_0$. Because $\tilde{U}_0$ a product, ~~hence~~, $(*)$ implies $F \circ \varphi^{-1}(\tilde{U}_0) \subseteq V_0$. Define coordinates

$$\psi : V_0 \to \mathbb{R}^n$$
$$(v,w) \mapsto (v, w - S(v)),$$

which (as needed to be a chart) is a diffeo onto its image because $\psi^{-1}(s,t) = (s, t + S(s))$.

Finally,

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi(x, S(x)) = (x, S(x) - S(x)) = (x,0).$$

$\square$

Theorem 4.14 Let $F : M \to N$ be smooth of constant rank.

(a) $F$ surjective $\Rightarrow$ $F$ submersion.

(b) $F$ injective $\Rightarrow$ $F$ immersion

(c) $F$ bijective $\Rightarrow$ $F$ diffeomorphism.

Proof (c) follows from (a) and (b).

(b) follows from rank theorem

(a) follows " " " + Baire category theorem $\square$

EOC 9/11/24

# Embeddings.

**Def** $F: M \to N$ is a <u>smooth embedding</u> if.
(i) $F$ is a smooth immersion, and
(ii) $F$ is a homeomorphism onto its image.

<u>Examples</u>
- ~~open~~ inclusions of open submanifolds (easy)
- inclusion $S^n \to \mathbb{R}^{n+1}$ (why??)

<u>Non-examples</u>

- $\gamma: \mathbb{R} \to \mathbb{R}^2$
  $t \mapsto (t^3, 0)$

- $\beta: (-\pi, \pi) \to \mathbb{R}^2$
  $t \mapsto (\sin 2t, \sin t)$

- irrational slopes on torus. (It is an immersion.)

<u>Thm 4.24</u> Immersion $\Leftrightarrow$ local embedding.

**Proof** $\Leftarrow$: obvious
$\Rightarrow$: rank thm/IFT implies "local injection." But irrational slope on a torus is a local injection that isn't a ~~smooth~~ (topological) embedding. If we look harder at rank theorem and

pick coordinates ^domain^ around a point small enough, ~~so that~~ there nothing funny happens. $\square$

# Lecture 4.3

__Reading__: Start Ch. 5

__Practice__: 5.10

__Homework__: 5-4

Note: "Embeddings" stuff on previous page will need to be covered after submersions.

## __Properties of Submersions__

__Thm 4.26__ (Local Section Theorem)

Suppose $\pi: M \to N$ smooth. Then $\pi$ is a submersion iff every point of $M$ is in the image of a __smooth local section__.

__Def__ If $\pi: M \to N$ is any map, a __smooth section__ is a smooth $s: N \to M$ such that $\pi \circ s = \mathrm{Id}_N$. (Right inverse.) A __local section__ of $\pi$ is a smooth map $s: U \to M$ with $\pi \circ s = \mathrm{Id}_U$ for some open $U \subseteq N$.

__Proof of Thm 4.26:__

Use rank theorem and make following picture precise.