pick coordinates domain around a point small enough, ~~so that~~ there nothing funny happens. $\square$

Lecture 4.3

Reading: Start Ch. 5

Practice: 5.10

Homework: 5-4

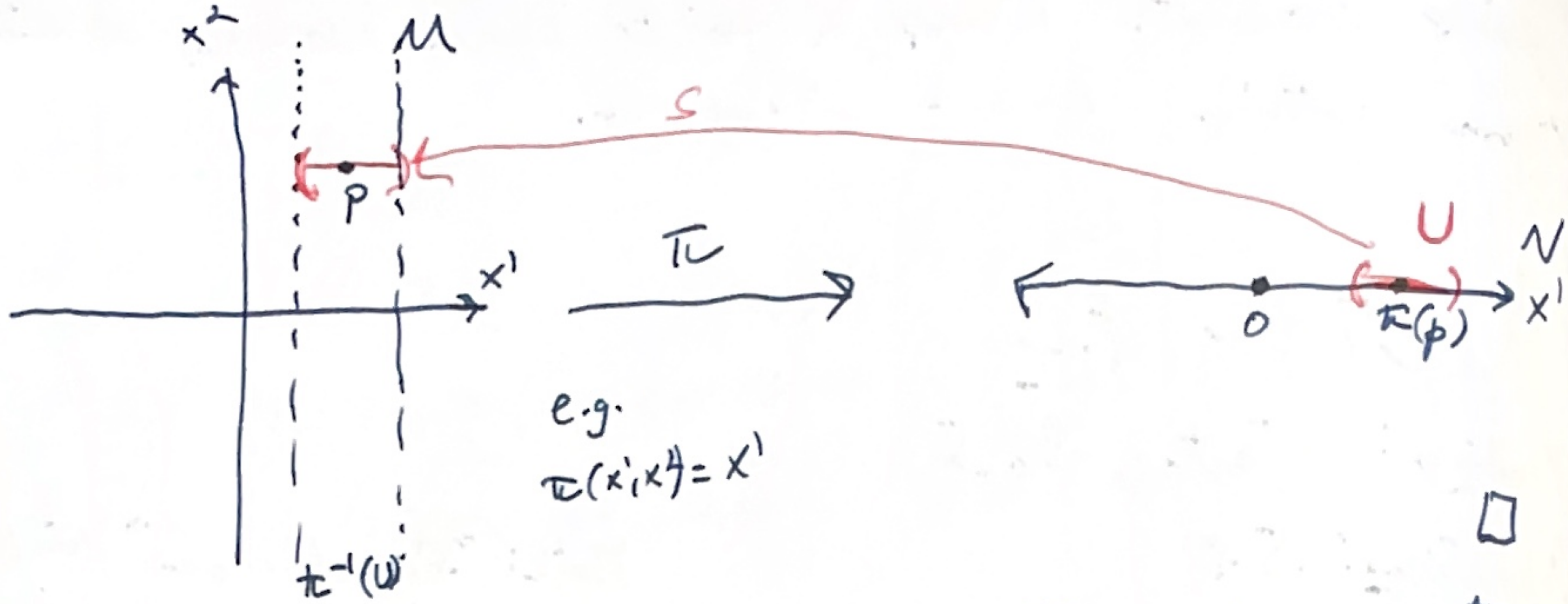
Note: "Embeddings" stuff on previous page will need to be covered after submersions.

Properties of Submersions

Thm 4.26 (Local Section Theorem)
Suppose $\pi: M \to N$ smooth. Then π is a submersion iff every point of M is in the image of a smooth local section.

Def If $\pi: M \to N$ is any map, a smooth section is a smooth $s: N \to M$ such that $\pi \circ s = \mathrm{Id}_N$. (Right inverse) A local section of π is a smooth map $s: U \to M$ with $\pi \circ s = \mathrm{Id}_U$ for some open $U \subseteq N$.

Proof of Thm 4.26:
Use rank theorem and make following picture precise.



e.g.
$\pi(x^1, x^2) = x^1$

$\square$

<u>Corollary 4.29</u> (Universal property of surjective submersions) Let $\pi : M \to N$ be a surjective submersion. Given any smooth P, a map $F : N \to P$ is smooth iff $F \circ \pi$ is smooth.

$\square$

<u>Proof</u>: $\Rightarrow$: obvious. $\Leftarrow$: use local section. $\square$

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F\circ\pi"] & \\ N \arrow[r, "F"'] & P \end{tikzcd}$$

Need to use this property to ~~p~~ solve Problem 4-7.

<u>Embeddings</u> section on previous pages.

<u>Embedded submanifolds</u>

<u>Def</u> Let M be a smooth manifold. An <u>embedded submanifold</u> is a subspace $S \subseteq M$ that is a (topological) manifold, endowed with a smooth structure making the inclusion map $S \hookrightarrow M$ a smooth embedding.

The <u>codimension</u> of S is $\dim M - \dim S$, and M is sometimes called the <u>ambient manifold</u>

<u>Prop 5.1</u>

"codimension 0 embedded submanifold of M" = "open submanifold"

<u>Prop 5.2</u>

If $F: N \to M$ is a smooth embedding, then $S = F(N)$ is ~~has~~ a unique smooth structure satisfying two properties:

(i) S is an embedded submanifold

(ii) $F: N \to M$ is a diffeomorphism onto its image.

Neither proposition has a difficult proof, see book.

<u>Def</u> a <u>level set</u> of $\Phi: M \to N$ is any subset of form $\Phi^{-1}(c) \subseteq M$ for some $c \in N$. (If $N = \mathbb{R}^n$ and $0 = c$, call this the <u>zero set</u> of Φ.)

<u>Theorem</u> 5.12 (Level sets of constant rank maps)

Let $\Phi: M \to N$ be a smooth map of constant rank r. Then each level set is a (properly) embedded submanifold of M of codimension r.

Proof Let $\dim M = m$, $\dim N = n$, $k = m - r$, $c \in N$ and $S = \Phi^{-1}(c) \subseteq M$. We will use rank theorem to build an atlas on S. Given $p \in S$, Pick coordinates (U, φ) centered at p and (V, ψ) centered at $\Phi(p) = c$ in which

$$\hat{\Phi}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r, \underbrace{0, \dots, 0}_{n-r}).$$

Notice that $\varphi(S \cap U)$ is the "slice"

$$\varphi(S \cap U) = \{ (\underbrace{0, \dots, 0}_{r}, x^{r+1}, \dots x^m) \in \varphi(U) \} \subseteq \mathbb{R}^m$$

Define $\tilde{\varphi} : S \cap U \to \tilde{\varphi}(S \cap U) \subseteq \mathbb{R}^{m-r}$

$$q \mapsto \pi \circ \varphi(q)$$

where $\pi : \mathbb{R}^m \to \mathbb{R}^{m-r}$, $(x^1, \dots, x^m) \mapsto (x^{r+1}, \dots, x^m)$. Can argue that the family $\{(S \cap U, \tilde{\varphi})\}_{p \in S}$ (where $\{(U, \varphi)\}_{p \in M}$ are charts coming from rank theorem) gives a smooth structure on S. Why? Let

$j : \mathbb{R}^{m-r} \to \mathbb{R}^m$, $(y^1, \dots, y^{m-r}) \mapsto (0, \dots, 0, y^1, \dots, y^{m-r})$, which is clearly smooth. Then the transition functions are

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} : \tilde{\varphi}(S \cap U \cap V) \to \tilde{\psi}(S \cap U \cap V)$$

which we can identify with

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} = \pi \circ \psi \circ \varphi^{-1} \circ j,$$ which is clearly smooth.

Thus, S admits a smooth structure. It's clearly Hausdorff + 2nd countable, since it's a subspace of M.

Let's show the inclusion $S \hookrightarrow M$ is an embedding.

Well, use the "obvious" coordinates: given $p \in S$, let (U, φ) be a chart coming from rank theorem for Φ, as above. Then in the $(S \cap U, \tilde{\varphi})$ coordinates on S and the (U, φ) chart on M, have

$$i : S \hookrightarrow M$$
$$(y^1, \dots, y^{m-r}) \mapsto (0, \dots, 0, y^1, \dots, y^{m-r}).$$ $\square$

Lecture 5.1

<u>Reading</u>: Middle of Ch. 5

<u>Practice</u>: 5.20

<u>Homework</u>:

Begin by stating and proving <u>Thm 5.12</u> on previous page.

<u>Corollary 5.13</u> If $\Phi : M \to N$ a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension equals $\dim N$.