which we can identify with

$\tilde{\psi} \circ \tilde{\varphi}^{-1} = \pi \circ \psi \circ \varphi^{-1} \circ j$, which is clearly smooth.

Thus, S admits a smooth structure. It's clearly Hausdorff + 2nd countable, since it's a subspace of M.

Let's show the inclusion $S \hookrightarrow M$ is an embedding.

Well, use the "obvious" coordinates: given $p \in S$, let (U, φ) be a chart coming from rank theorem for Φ, as above. Then in the $(S \cap U, \tilde{\varphi})$ coordinates on S and the (U, φ) chart on M, have

$$i : S \hookrightarrow M$$
$$(y^1, \dots, y^{m-r}) \mapsto (0, \dots, 0, y^1, \dots, y^{m-r}).$$

$\square$

Lecture 5.1

<u>Reading</u>: Middle of Ch. 5

<u>Practice</u>: 5.20

<u>Homework</u>:

Begin by stating and proving <u>Thm 5.12</u> on previous page.

<u>Corollary 5.13</u> If $\Phi : M \to N$ a smooth submersion, then each level set of Φ is a properly embedded submanifold whose codimension equals $\dim N$.

Remarks

- Corollary 5.13 can be strengthed: smooth submersions induce foliations on their domains.
- Theorem 5.12 (and its proof) can be generalized to characterize embedded submanifolds as those subsets admitting "slice charts." See Theorem 5.8.

We're now going to generalize Corollary 5.13, based on the observation that if $m \geq n$, then the set of points ~~in~~ $p \in M$ such that $\Phi : M \to N$ has $d\Phi_p$ full rank, is open in M.

Def $p \in M$ is a regular point of $\Phi : M \to N$ if $d\Phi_p$ is surjective, and otherwise called a critical point.

Def $c \in N$ is a regular value of $\Phi : M \to N$ if every $p \in \Phi^{-1}(c)$ is a regular point, and otherwise c is called a critical ~~point~~ value.

Def $\Phi^{-1}(c)$ is a regular level set of $\Phi : M \to N$ if c is a regular value of Φ.

<u>Corollary 5.14</u> Every regular level set of every smooth map ~~of~~ is an (properly) embedded submanifold of the domain of codimension equal to the dimension of the codomain.

<u>Proof</u> Let c be a regular value of $\Phi : M \to N$ and let $U \subseteq M$ be the open set consisting of full rank points. Apply Corollary 5.13 to $\Phi|_U$, which, by construction, is a submersion. $\square$

<u>Example</u> $S^n \subseteq \mathbb{R}^{n+1}$ is properly embedded because it is the regular level set of

$$\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n \qquad \Phi^{-1}(1) = S^n.$$
$$x \mapsto |x|^2$$

<u>Remark</u> <u>Morse theory</u> studies manifolds by using smooth functions $F : M^m \to \mathbb{R}$ and analyzing what happens to level sets at critical values. (Usually restrict to "Morse" F, meaning the Hessian $\left(\frac{\partial^2 F}{\partial x^i \partial x^j}\Big|_p\right)_{i,j=1}^m$ is assumed to be non-singular at all critical points p. This guarantees all kinds of nice things, e.g. that critical points are isolated.)

Immersed Submanifolds

<u>Def</u> An <u>immersed submanifold</u> of M is a subset S endowed with a topology ~~and~~ (not necessarily subspace topology) w.r.t. which S is a topological manifold, together with a smooth structure making $i: S \hookrightarrow M$ a smooth immersion.

<u>Prop 5.18</u> Let $F: N \to M$ be an <u>injective</u> immersion and $S = F(N)$. Then S has a unique topology and smooth structure making it an immersed submanifolds.

<u>Remark</u>: one could more generally define immersed submanifolds as images of (~~not~~-necessarily injective) immersions. Lee does not do this.