Lecture 5.2

Reading: Finish Ch. 5

Practice: 5.36, 5.40

Homework: 5-6, 5-7, 5-19 (HW5)

Restriction of Smooth Maps

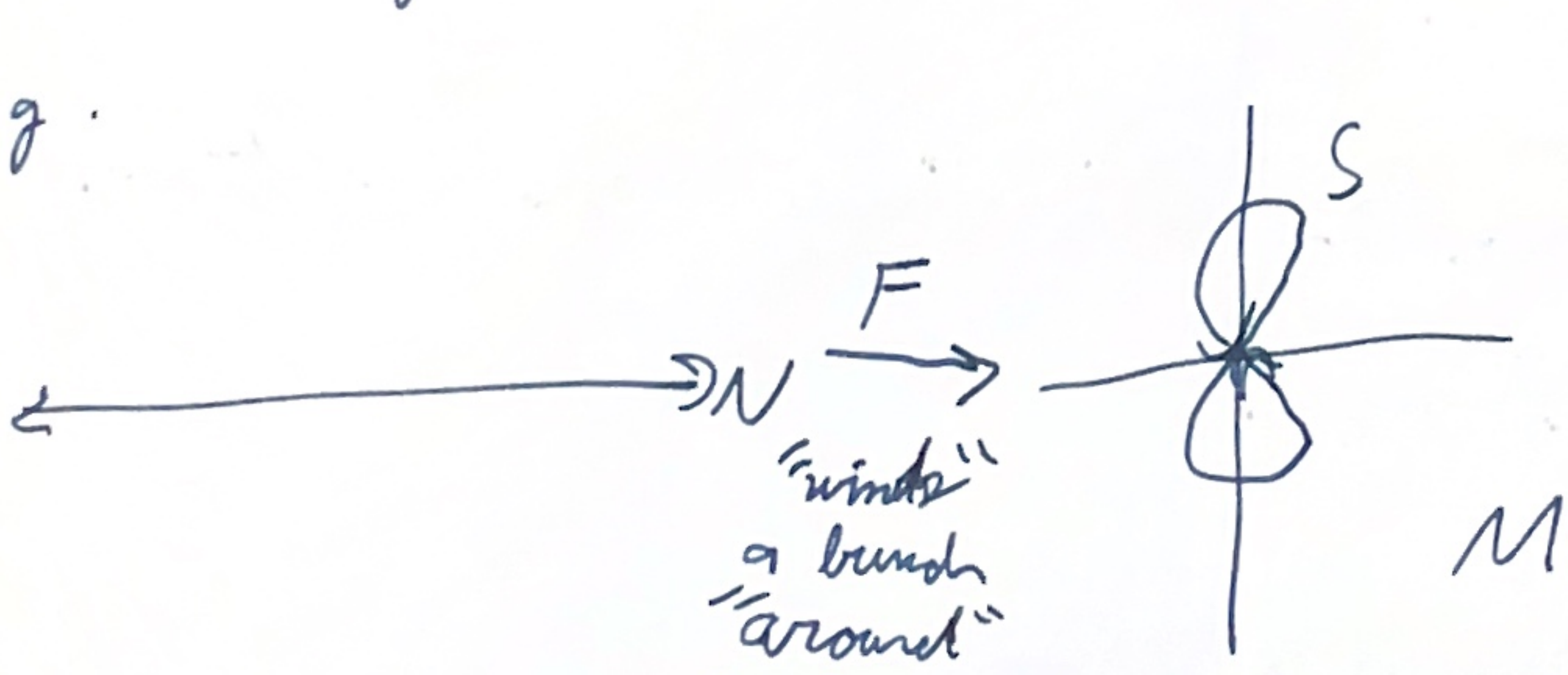
Theorem 5.27 If $F: M \to N$ smooth and $S \subseteq M$ is immersed or embedded, then $F|_S$ is smooth.

Proof: Obvious: $F|_S = F \circ \iota$, where $\iota: S \hookrightarrow M$ smooth. $\square$

Theorem 5.29 Suppose $F: N \to M$ smooth and $S \subseteq M$ immersed. If $F(N) \subseteq S$ and $F: N \to S$ is continuous (w.r.t. whatever topology is on S as an immersed submanifold), then $F: N \to S$ is smooth

Note: I view the hypothesis that "$F: N \to S$ is continuous" as a failure of imagination related to Lee's restricted definition of "immersed submanifold."

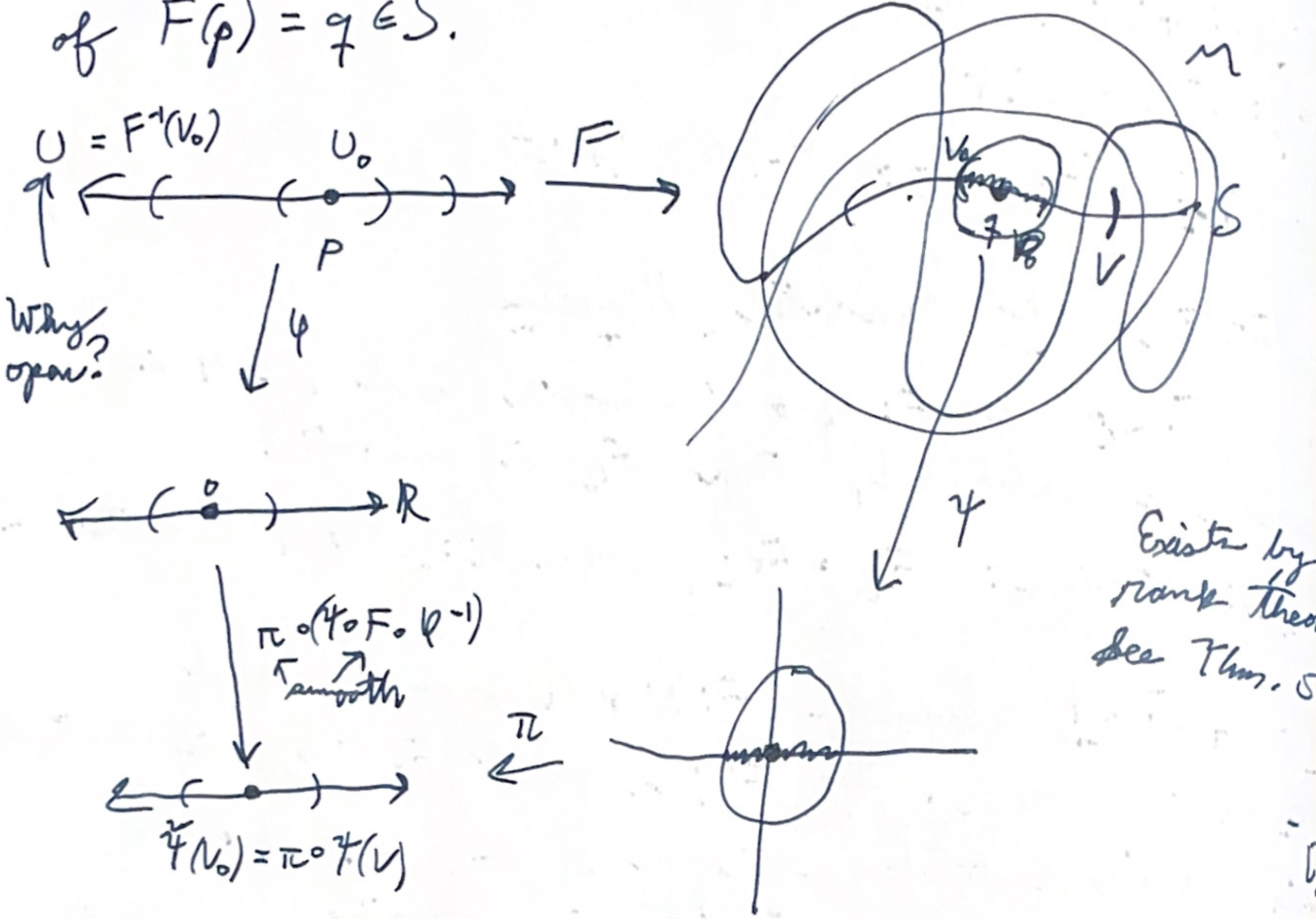
E.g.



S... is not continuous!

Proof: Use slice chart (V_0, ψ) for an open neighborhood V of $F(p) = q \in S$.

Why open?



Exists by rank theorem See Thm. 5.8. ∎

<u>Corollary 5.30</u> If $S \subseteq M$ is immersed submanifold with the subspace topology, ~~...~~ and $F: N \to M$ is smooth with $F(N) \subseteq S$, then $F: N \to S$ smooth. ∎

Why? One-line...

Ex S embedded, or S irrational slope on torus.

Using this, can prove

Thm 5.31 Suppose $S \subseteq M$ is a smoothly embedded submanifold. Then there exist exactly one topology on S (namely, subspace topology) and smooth structure (namely, slice charts) that make S immersed.

Thm 5.32 Suppose $S \subseteq M$ is immersed. Then for the given topology on S, there is exactly one smooth structure (the given one, namely) making S immersed.

Ex

Restricting Tangent Spaces

Note: if $S \subseteq M$ immersed, get a canonical injection

$$di_p : T_pS \hookrightarrow T_pM$$

for all $p \in S$. We will thus conflate T_pS with its image under di_p.

Prop 5.37 Suppose $S \subseteq M$ embedded and $p \in S$.

Then

$$T_p S = \{ v \in T_p M \mid vf = 0 \quad \forall f \in C^\infty(M) \text{ s.t. } f|_S \equiv 0 \}.$$

Proof $\subseteq$: easy. $\supseteq$: Use slice coordinates (and a bump function supported on the domain) $\square$

Ex 5.40: If $S = \Phi^{-1}(c) \subseteq M$ a level set of constant rank map $\Phi : M \to N$, then $T_p S = \ker d\Phi_p$.