Prop 5.37 Suppose $S \subseteq M$ embedded and $p \in S$.
Then
$$T_pS = \{v \in T_pM \mid vf = 0 \;\; \forall f \in C^\infty(M) \text{ s.t. } f|_S \equiv 0\}.$$

Proof $\subseteq$: easy. $\supseteq$: Use slice coordinates (and a bump function supported on the domain) $\square$

Ex 5.40: If $S = \Phi^{-1}(c) \subseteq M$ a level set of constant rank map $\Phi : M \to N$, then $T_pS = \ker d\Phi_p$.

Lecture 5.3

Reading: First half of Ch. 6
Practice: n/a **Homework**: 6-2

Today we prove

Thm 6.15 (Easy Whitney Embedding)
Every smooth n-manifold admits a (proper) smooth embedding in $\mathbb{R}^{2n+1}$.

In fact, more is true, but the proof is harder:

Thm (Strong Whitney Embedding)
Every smooth n-manifold (properly) smoothly embeds in $\mathbb{R}^{2n}$.

This harder result's proof employees the "Whitney trick," which promotes an immersion in $\mathbb{R}^{2n}$ to an embedding. This trick was ~~~~ employed later by Smale to prove the h-cobordism theorem, which implies ~~the~~ certain cases of generalized Poincaré conjecture in $d \geq 5$ eventually.

On HW6, you'll prove

<u>Thm 6.18</u> (Easy Whitney Immersion Theorem)
Every smooth n-manifold smoothly immerses into $\mathbb{R}^{2n}$.

Strong immersion: $\mathbb{R}^{2n-1}$.

<u>Note</u>: Even strong versions of theorems not sharp for all n. E.g. 3-manifolds always embed in $\mathbb{R}^5$. ~~4-manifolds do~~ However they are sharp in infinitely many dimensions, e.g. $\mathbb{RP}^{2^n}$ cannot embed in $\mathbb{R}^d$ for $d < 2 \cdot 2^n = 2^{n+1}$.

For sake of time, in class today we will only prove the weak/easy embedding theorem for compact manifolds. Compactness makes life easier for the following reason:

Proposition If M is compact and $F: M \to N$ is an injective smooth immersion, then F is a smooth embedding.

Proof If M is compact, then ~~(b/c N Hausdorff)~~ then F is closed. (Exercise in point-set topology... need to use that N is Hausdorff.) Thus F is a topological embedding that is smooth, meaning F is a smooth embedding. $\square$

~~This~~ compactness is a helpful simplifying assumption but we still need a big tool.

Sard's theorem

Def $A \subseteq \mathbb{R}^n$ has zero measure if for any $\delta > 0$, A can be covered by a (countable) union of open balls, the sum of whose volumes is less than δ.

Def If M^n a smooth manifold, then $A \subseteq M$ has measure zero in M if for every smooth chart (U, φ) on M, $\varphi(A \cap U)$ has zero measure in $\mathbb{R}^n$.

Note: If A has measure zero, then $M - A$ is everywhere dense. Even better: $M - A$ has "full measure"

Note that pre-images of measure zero sets under smooth maps need not be measure zero. However, images of measure zero sets are measure zero. (Thm 6.9)

Theorem 6.10 (Sard's Theorem)
If $F: M \to N$ smooth, then the set of critical values has zero measure.

Proof of Easy Whitney Embedding
When M compact, theorem will follow from following two lemmas (and the proposition above).

Lemma 1 If M is a compact smooth manifold, then there is some $N > 0$ such that M injectively immerses (hence, embeds) into $\mathbb{R}^N$.

Lemma 2 Suppose $M \subseteq \mathbb{R}^N$ is a smoothly immersed n-dimensional submanifold. For any $v \in \mathbb{R}^N - \mathbb{R}^{N-1}$, let $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1}$ be the (non-orthogonal!) projection with kernel $\mathbb{R}v$. If $N > 2n+1$, then there is a dense set of $v \in \mathbb{R}^N - \mathbb{R}^{N-1}$ for which $\pi_v|_M$ is an injective immersion into $\mathbb{R}^{N-1}$.

As we will see, only Lemma 2 needs Sard's theorem.

Proof of Lemma 1:

Since M^n is compact, we can find a finite cover by regular coordinate balls $B_1, \dots, B_m$. That is, for each B_i, exists $B_i' \supseteq \overline{B_i}$ and $\varphi_i : B_i' \to \varphi(B_i') \subseteq \mathbb{R}^n$ that takes $\overline{B_i}$ to $\overline{\mathbb{B}_n}$. For each i, let $\rho_i : M \to \mathbb{R}$ be a bump function for $\overline{B_i}$ supported in B_i'. Now define

$$F : M \to \mathbb{R}^{nm+m}$$
$$p \mapsto (\rho_1(p)\varphi_1(p), \dots, \rho_m(p)\varphi_m(p), \rho_1(p), \dots, \rho_m(p)).$$

We claim F is an injective smooth immersion.

Injective: if $p, q \in B_i$ for same i, then $F(p) \neq F(q)$ b/c $\rho_i(p)\varphi_i(p) = \varphi_i(p) \neq \varphi_i(q) = \rho_i(q)\varphi_i(q)$.

if $p \in B_i$, $q \notin B_i$ for some i, then $F(p) \neq F(q)$ b/c $\rho_i(p) = 1 \neq \rho_i(q)$.

Smooth: clear.

Immersion: for any $p \in M$, $\exists i$ with $p \in B_i$. Then dF_p has a "slab" that looks like $d(\rho_i(p)\varphi_i(p))|_p = d\varphi_i|_p$, which is injective, hence so is dF_p. $\square$

__Proof of Lemma 2:__ Note:

$\pi_v|_M$ injective $\Leftrightarrow$ $p-q$ is not parallel to v for all $p \neq q \in M$.

$\pi_v|_M$ smooth immersion $\Leftrightarrow$ $T_pM \subseteq T_p\mathbb{R}^N = \mathbb{R}^N$ contains no vectors parallel to v.

Define

$$\Delta_M = \{(p,p) \mid p \in M\} \subseteq M \times M \quad (\Delta_M = \Gamma_{Id_M})$$

$$M_0 = \{(p,0) \mid p \in M,\ 0 \in T_pM\} \subseteq TM \quad \text{"zero section of } TM\text{"}$$

Consider smooth maps

$$K : M \times M - \Delta_M \to \mathbb{RP}^{N-1}$$
$$(p,q) \mapsto [p-q]$$

$$\tau : TM - M_0 \to \mathbb{RP}^{N-1}$$
$$(p,w) \mapsto [w].$$

Remember: $M \subseteq \mathbb{R}^N$, so it makes sense to write $[p-q]$ and to consider $w \in T_pM$ as a vector in $T_p\mathbb{R}^N \cong \mathbb{R}^N$

v is injective smooth immersion iff $v \notin \operatorname{Im} K$ and $v \notin \operatorname{Im} \tau$.

$$\dim (M \times M - \Delta_M) = \dim (TM - M_0) = 2n < N-1 = \dim \mathbb{RP}^{N-1}$$

$\xRightarrow{\text{Sard}}$ $\operatorname{Im} K \cup \operatorname{Im} \tau$ is measure zero in $\mathbb{RP}^{N-1}$!

$\Rightarrow \exists$ dense set of $[v] \in \mathbb{RP}^{N-1}$ where $\pi_v|_M$ is an injective immersion. $\square$