# Lecture 6.1

<u>Reading</u>: Second half of Ch. 6

<u>Homework</u>: 6-5, 6-10

Finish proof of Lemma 2 on previous page.

<u>Tubular Neighborhoods</u>

Given an embedded submanifold $M^m \subseteq \mathbb{R}^n$, we'd like to understand "nice" open neighborhoods of $M$ in $\mathbb{R}^n$.

<u>Def</u> Given embedded submanifold $M^m \subseteq \mathbb{R}^n$, and $p \in M$, the <u>normal space</u> to $M$ at $p$ is

$$N_p M = \{ v \in T_p\mathbb{R}^n \mid v \cdot w = 0 \quad \forall w \in T_p M \subseteq T_p \mathbb{R}^n \}.$$

The <u>normal bundle</u> is the set

$$NM = \{ (p, v) \in M \times \mathbb{R}^n \mid v \in N_p M \},$$

together with natural projection map

$$\pi : NM \to M.$$

<u>Thm 6.23</u> If $M^m \subseteq \mathbb{R}^n$ embedded, then $NM$ is an embedded $n$-dimensional submanifold of $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

<u>Idea of proof</u>: Can promote a slice chart for $M$ in $\mathbb{R}^n$ to a slice chart for $NM$ in $T\mathbb{R}^n$.

Let $E: NM \to \mathbb{R}^n$.
$(p,v) \mapsto p+v$

Note $E$ is smooth, but almost never injective. $E(p,0) = p = \mathrm{Id}_M$

**Def**: A <u>tubular neighborhood of $M \subseteq \mathbb{R}^n$</u> is an open neighborhood $U$ of $M$ in $\mathbb{R}^n$ that is the diffeomorphic image of an open subset $V \subseteq NM$ of the form

$$V = \{(p,v) \mid |v| < \delta(p)\}$$

for some ~~xx~~ positive ~~smooth~~ continuous function $\delta : M \to \mathbb{R}$.

<u>Thm 6.24</u> Every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.

For sake of time, I am skipping the proof.
We like tubular neighborhoods for following reason.

**Def** If $M \subseteq X$ a subspace, then $r : X \to M$ is a <u>retraction</u> if $r|_M = \mathrm{Id}_M$.

<u>Prop 6.25</u> If $U$ is a tubular neighborhood of the embedded submanifold $M \subseteq \mathbb{R}^n$, then there exists a smooth retraction $r : U \to M$ that is a submersion.

# Transversality

a kind of generalization of "submersion" relative to embedded submanifolds.

**Def** Two embedded submanifolds $S, S' \subseteq M$ intersect transversely if for each $p \in S \cap S'$,

$$T_p M = \text{span}\{T_p S, T_p S'\}.$$

**Def** If $F: N \to M$ smooth and $S \subseteq M$ embedded, then we say F is transverse to S if for all $x \in F^{-1}(S)$

$$T_{F(x)} M = \text{span}\{T_{F(x)} S, dF_x(T_x N)\}.$$

Examples: - two distinct points in $\mathbb{R}^2$

- two lines in $\mathbb{R}^2$
- one line and one plane in $\mathbb{R}^3$

Non-examples

- $S = p \in \mathbb{R}^2$, $S'$ a line passing through $p$.
- two lines in $\mathbb{R}^3$ that intersect.
- $S = \{(x, x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, $S' = \{(x, -x^2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

Note: TFAE

- $F: M \to N$ is a submersion
- ~~$\forall p \in N$~~, $F$ is transverse to ~~$p$~~ $S = N$
- $\forall$ embedded $S \subseteq N$, $F$ is transverse to $S$.

This makes my interpretation of transversality as a "relative" submersion more precision.

A big reason we like submersions is that their level sets are always embedded submanifolds of known dimension. This generalizes to transversality as follows:

Thm 6.30

(a) If $F: N \to M$ smooth and transverse to $S \subseteq M$, then $F^{-1}(S)$ is an embedded submanifold of $M$ whose codimension is equal to the codimension of $S$ in $M$.

(b) If $S, S' \subseteq M$ are transverse embedded submanifolds, then $S \cap S'$ is an embedded submanifold whose codimension is the sum of the codimensions of $S$ and $S'$.

You should read the proof on your own time.

A big reason transversality is even nicer than submersion, formally speaking, is that it is a "generic" property of "families" of smooth functions. Can make this precise in various ways.

## Parametric Transversality Theorem

Let $N, M, S$ be smooth manifolds and $X \subseteq M$ embedded. If

$$F : N \times S \to M$$

is transverse to $X$, then

$$F_s : N \to M$$
$$p \mapsto F(p, s)$$

is transverse to $X$ for almost all $s \in S$.

Intuition: Most "slices" of a smooth function transverse to $X \subseteq M$ are still transverse to $X$.

Proof is an application of Sard.

## Transversality ~~Homotopy~~ Theorem

If $f : N \to M$ smooth and ~~transverse to~~ $X \subseteq M$, embedded

then there exists a smooth function

$$F : ~~N \to M~~ \; N \times [0,1] \to M$$

such that $F(p, 0) = f(p)$ and ~~$g(p) := F(p,1)$ is~~

$$g : N \to M$$
$$p \mapsto F(p,1)$$

is transverse to $X$.