Lecture 6.3

Reading: First three sections of Ch. 7

Practice: 7.2, 7.13, 7.20, 7-7

Homework: 7-2, 7-4.

Announcement: Q2 next week Friday or Wednesday?

Lie Groups

Def A *Lie group* is a smooth manifold G that is also a group for which multiplication

$$m : G \times G \to G$$
$$(g, h) \mapsto gh$$

and inversion

$$i : G \to G$$
$$g \mapsto g^{-1}$$

are smooth maps.

Given a (Lie) group G and $g \in G$, write

$$L_g : G \to G, \quad h \mapsto gh$$

and

$$R_g : G \to G$$
$$h \mapsto hg$$

These are diffeomorphisms.

Examples

- $(\mathbb{R},+)$, $(\mathbb{R}^n,+)$, $(S^1 \subseteq \mathbb{C}^*, \cdot)$, $(\mathbb{R}^* = \mathbb{R} - \{0\}, \cdot)$
- $GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$ with matrix multiplication. Related:

  $GL^+(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A > 0\}$

  $SL(n,\mathbb{R}) = \{A \in GL^+(n,\mathbb{R}) \mid \det A = 1\}$

  $O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid A^T = A^{-1}\}$

  $SO(n,\mathbb{R}) = \{A \in O(n,\mathbb{R}) \mid \det A = 1\} \subseteq SL(n,\mathbb{R})$
- $(\mathbb{C},+)$ is a 2-dimensional Lie group isomorphic to $(\mathbb{R}^2,+)$
- $(\mathbb{C}^*,\cdot)$
- Products of Lie groups, e.g.

  $$\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$$
- Any finite group equipped with discrete topology is a 0-dimensional (disconnected) Lie group.

Def If G and H are Lie groups, a <u>(Lie group) homomorphism</u> is a smooth map $F: G \to H$ that is also a group homomorphism. If F is a diffeomorphism, then we call it a <u>(Lie group) isomorphism</u>.

<u>Examples</u>

- all of the implicitly suggested inclusions on previous page
- $\exp: \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$, $\exp: \mathbb{R} \to \mathbb{R}^*$, $t \mapsto e^t$
- Fix $g \in G$ and consider <u>conjugation by g</u>:

$$C_g: G \to G$$
$$h \mapsto ghg^{-1}.$$

This is a (Lie group) isomorphism.

<u>Theorem 7.5</u> Every Lie group homomorphism.

<u>Proof</u>: Given $F: G \to H$ and $g \in G$, note that F is a homomorphism $\Rightarrow L_{F(g)} \circ F = F \circ L_g$. By chain rule:

$$d(L_{F(g)} \circ F)_e = d(F \circ L_g)_e$$

$$\Rightarrow \quad d(L_{F(g)})_{\tilde{e}} \circ dF_e = dF_g \circ (dL_g)_e$$

where $e \in G$ and $\tilde{e} \in H$ are identity elements. Since $d(L_{F(g)})_{\tilde{e}}$ and $(dL_g)_e$ are isomorphisms, we conclude that

$$\operatorname{rank} dF_g = \operatorname{rank} dF_e. \qquad \square$$

Applying global rank theorem gets us:

Corollary A Lie group homomorphism is a Lie group isomorphism iff it is bijective.

Def A Lie subgroup of G is an immersed submanifold ~~for~~ ~~it~~ that is also a subgroup.

Prop 7.11 If $H \leq G$ is a subgroup that is an embedded submanifold, then H is automatically a Lie subgroup.

Proof It's clear that $m|_{H\times H} : H\times H \to H$ and $i|_H : H \to H$ are smooth iff they are continuous. Well, they are, because H is embedded. $\square$

Lemma 7.2 Any open submanifold of a Lie group G ~~that~~ that is a Lie subgroup is also closed; hence, open Lie subgroups are always unions of connected components of G.

PF: $G - H = \bigcup_{g \in G-H} gH$ where each $gH = L_g(H)$ is open because L_g a diffeomorphism.

$\Rightarrow G - H$ ~~is~~ open $\Rightarrow H$ closed. $\square$

Ex If G connected, it only has one open Lie subgroup (all of G). How many open Lie subgroups does $GL(n,\mathbb{R})$ have?

Def The identity component of G is the connected component containing e. Denoted G_0.

Prop 7.15 G_0 is a normal subgroup of G, and the only connected component that is a subgroup. Every connected component is diffeomorphic to G_0.

Prop 7.16

Proof: do it at end of class.

Prop 7.17: If $F: G \to H$ injective Lie grou