Lecture 7.1

Reading: First two sections of Ch. 8

Practice: 8.7, 8-5, 8.18

Homework: 8-1, 8-12, 8-13

Vector fields

Def If M a smooth manifold, a rough vector field is any ~~smooth~~ function $X : M \to TM$, $p \mapsto X_p$ such that $\pi \circ X = \mathrm{Id}_M$. A smooth vector field is a rough vector field X such that X is smooth.

If $(U, (x^i))$ smooth chart, can write

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

where $X^i :$ ~~[illegible]~~ $U \to \mathbb{R}$ are components of X in the given chart.

Prop 8.1 A (rough) vector field is smooth iff its components are.

Proof: Look at natural coordinates on TM. $\square$

$\underline{\text{Ex}}$ $\cancel{\mathbb{R}^n}$ $\mathbb{R}^n$ $\quad p \mapsto \frac{\partial}{\partial x^i}\Big|_p$ (coordinate vector fields)

for any $i = 1, \dots, n$.

$$V : \mathbb{R}^n \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \qquad \text{(Euler vector field)}$$
$$p \mapsto (p, p) = \sum_{i=1}^n p^i \frac{\partial}{\partial x^i}\Big|_p$$

- ~~[illegible]~~ (U, φ) a chart on M: can define v.f. on U by $p \mapsto \frac{\partial}{\partial x^i}\Big|_p$
- Let θ be any angle coordinate on S^1, and let $\frac{\partial}{\partial \theta}$ be corresponding coordinate v.f. Of course, a priori, depends on domain of θ and specific parametrization. However, if $\tilde{\theta}$ is any other angle coordinate, have $\frac{d}{d\tilde{\theta}} = \frac{d}{d\theta}$ on their overlap.

Why?

Thus, there is a globally defined smooth vector field that is nowhere vanishing.

$\underline{\text{Def}}$ If M smooth and $A \subseteq M$ any subset, a (rough) ~~smooth~~ vector field along A is a map $X : A \to TM$ such that $\pi \circ X = \mathrm{Id}_A$. We call X <u>smooth</u> along A if for each $p \in A$, there exists a neighborhood of p in M that supports a smooth extension of X.

On homework, you ~~..~~ should use partitions of unity to prove:

<u>Lem 8.6</u> If $A \subseteq M$ is closed, X a vector field along A and $U \supseteq A$ open neighborhood, then exists a smooth extension of X to M supported in U.

Taking $A = \{p\}$, get:

<u>Prop 8.7</u> Given $v \in T_p M$, there exists a smooth vector field X on M with $X_p = v$.

Note that it makes sense to add and scalar multiply two vector fields:

$$(aX + bY)_p = aX_p + bY_p.$$

Of course, there's also a 0 vector field. Thus

$$\mathfrak{X}(M) := \{\text{smooth vector fields } X : M \to TM\}$$

is an (infinite dimensional) vector space.

Even better: we can "scalar multiply" a vector field by any smooth function $f: M \to \mathbb{R}$:

$$(fX)_p = f(p) X_p.$$

(Why smooth again?) (use natural coordinates)

Prop 8.8 $\mathfrak{X}(M)$ is a module over $C^\infty(M)$.

Frames

Def a set of vector fields $X_1, \dots, X_k$ on M is called linearly independent if $\{X_1|_p, \dots, X_k|_p\}$ linearly independent for all ~~...~~ $p \in M$. They span the tangent bundle if $\{X_1|_p, \dots, X_k|_p\}$ spans T_pM $\forall p \in M$.

Def a local frame for M is an open neighborhood $U \subseteq M$ ~~...~~ and a set of linearly independent vector fields on U that span TU. If $U = M$, we have a global frame.

Ex The following admit (smooth) global frames:

$\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{S}^1$, $\mathbb{T}^n$, $\mathbb{C}$, every Lie group

Most manifolds do not admit a global frame. E.g.: $\mathbb{S}^2$ (hairy ball theorem).

Def M is parallelizable if it admits a (smooth) global frame.

Of course, every $p \in M$ ~~...~~ has a local frame.

See Prop. 8.11.

Derivations

M: C^∞ mfd. X: C^∞ v.f.

$$\Rightarrow X : C^\infty(M) \to C^\infty(M)$$
$$F \mapsto XF$$

where $(XF)(p) := X_p(F)$. Prop 8.14 shows XF is another smooth function on M.

Def A (global) derivation on M is a linear function ~~$X \in C^\infty$~~ $D : C^\infty(M) \to C^\infty(M)$ that is a derivation at every $p \in M$, i.e.

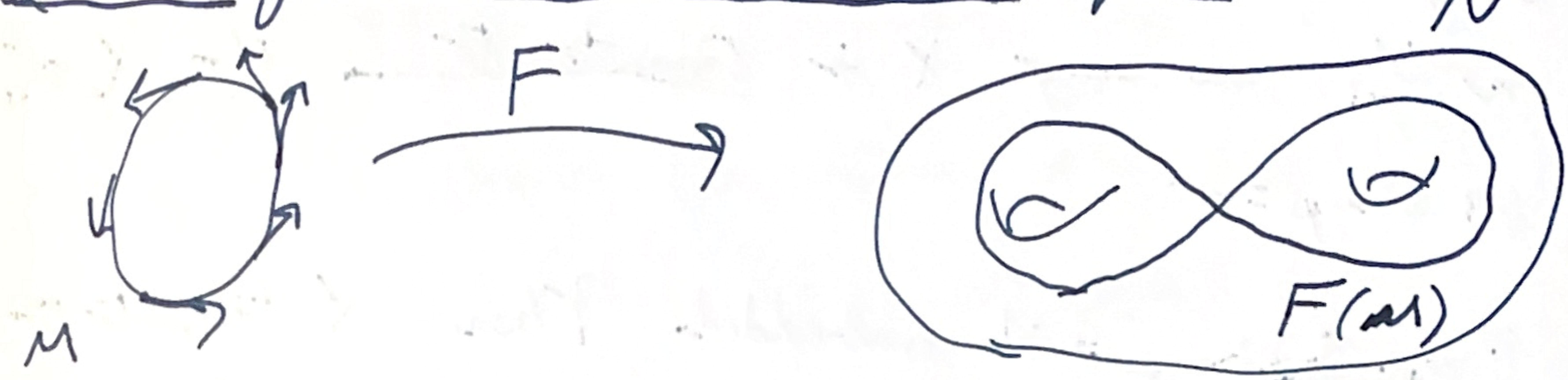
$$D(Fg)(p) = (DF)(p)\, g(p) + F(p)(Dg)(p).$$

Prop 8.15 $D : C^\infty(M) \to C^\infty(M)$ is a derivation iff exists a smooth vector field $X \in \mathfrak{X}(M)$ such that $DF = XF$ for all $F \in C^\infty(M)$. (In fact, X is unique.)

Proof idea Define X by

$$(X_p)(F) := (DF)(p). \qquad \square$$

Vector fields and smooth maps



Smooth map $F: M \to N$ lets us "pushforward" tangent vectors in M. However, we cannot typically pushforward vector fields.

Def For smooth $F: M \to N$ and vector field X on M, we say a vector field Y on N is F-related to X if $dF_p(X_p) = Y_{F(p)}$ $\forall p \in M$.

Prop 8.16 $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ F-related iff $X(f \circ F) = (Yf) \circ F$ $\forall f \in C^\infty(N)$.

Proof: $X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p) f$ and $(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)} f$

chain rule for directional derivatives

Prop 8.19 If $F: M \to N$ a diffeo., then for every $X \in \mathfrak{X}(M)$, there exists a unique smooth vector field on N that is F-related to X. Denote it $F_* X$.

In particular, $X(f \circ F) = \cancel{(F_*Xf) \circ F} \; (F_* X f) \circ F$

Def If $S \subseteq M$ immersed submanifold and X a v.f. on M, we say X is <u>tangent to S</u> if $X_p \in T_pS$ for all $p \in S$.

Skip in class

Prop. 8.22 $S \subseteq M$ embedded. Then X tangent to S iff $(XF)|_S = 0$ for every $F \in C^\infty(M)$ s.t. $F|_S \equiv 0$.

Prop 8.23 ...

Lecture 7.2

Reading: second half of Ch. 8

Practice: 8.29, 8.34, 8.35, 8.43

Homework: n/a

Begin w/ pushforwards, then derivations material on previous pages.

Lie brackets

Note that it makes sense to compose derivations. However, their composition is not usually a derivation!