Def If $S \subseteq M$ immersed submanifold and X a v.f on M, we say X is <u>tangent to S</u> iff $X_p \in T_pS$ for all $p \in S$.

skip in class

Prop. 8.22 $S \subseteq M$ embedded. Then X tangent to S iff $(XF)|_S = 0$ for every $F \in C^\infty(M)$ s.t. $F|_S \equiv 0$.

Prop 8.23 . . .

Lecture 7.2

Reading: second half of Ch. 8

Practice: 8.29, 8.34, 8.35, 8.43

Homework: n/a

Begin w/ pushforwards, then derivations material on previous pages.

Lie brackets

Note that it makes sense to compose derivations. However, their composition is not usually a derivation!

Ex $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial y}$ on $\mathbb{R}^2$, $f(x,y) = x$, $g(x,y) = y$.

Then $XY(fg) = 2x \neq x = fXYg + gXYf$.

"Correct" way to combine derivations is:

Def Let $X, Y \in \mathfrak{X}(M)$. The <u>(Lie) bracket</u> $[X, Y]$ is the operator

$$[X,Y] : C^\infty(M) \to C^\infty(M)$$
$$f \mapsto XYf - YXf.$$

<u>**Lemma**</u> $[X,Y]$ is a smooth vector field.

Pf Suffices to show $[X,Y]$ is a (global) derivation.

algebra... $\square$

<u>Remarks</u>

- In Ch. 9, we'll see that $[X,Y]$ is essentially the derivative of one vector field along the other
- Also in Ch. 9: $[X,Y] = 0$ iff flows commute
- $\mathfrak{X}(M)$ is essentially an infinite dimensional Lie algebra "over" $C^\infty(M)$ (rather than "just" $\mathbb{R}$).

Prop 8.26 If $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^i \frac{\partial}{\partial x^i}$ in coordinates (x^i), then in these coordinates

$$[X,Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}$$

$$= \left(X Y^j - Y X^j\right)\frac{\partial}{\partial x^j}$$ (abuse of notation)

Proof Of course

$$([X,Y]F)|_U = [X,Y](F|_U),$$

so we can work in a single chart.

$$[X,Y]F = X^i \frac{\partial}{\partial x^i}\left(Y^j \frac{\partial F}{\partial x^j}\right) - Y^j \frac{\partial}{\partial x^j}\left(X^i \frac{\partial F}{\partial x^i}\right)$$

$$= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial F}{\partial x^j} + X^i Y^j \frac{\partial^2 F}{\partial x^i \partial x^j} - \left(Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial F}{\partial x^i} + Y^j X^i \frac{\partial^2 F}{\partial x^j \partial x^i}\right)$$

$$= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial F}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial F}{\partial x^i})$$

$$= \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial F}{\partial x^j}.$$

$\square$

Prop 8.28

(a) $\forall a, b \in \mathbb{R}$, $[aX + bY, Z] = a[X, Z] + b[Y, Z]$

$$[X, aY + bZ] = a[X, Y] + b[X, Z]$$

(bilinear)

(b) $[Y, X] = -[X, Y]$. (anti-symmetric)

(c) (Jacobi)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

"derivative of associative identity"

(d) $\forall f, g \in C^\infty(M)$

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

There is also some naturality of Lie bracket under smooth maps: see Prop 8.30.

Lie algebras

Given a vector field X on Lie group G and an element $g \in G$, can concoct a new vector field $(L_g)_* X$ $(\cancel{= g \cdot X})$ by the formula

$$((L_g)_* X)_h = d(L_g)_{g^{-1}h}(X_{g^{-1}h})$$

Def X is a <u>left-invariant vector field on G</u> if

$(L_g)_* X = X$ for all $g \in G$.

<u>Prop 8.33</u> Let X, Y be smooth left invariant vector fields on G. Then $[X, Y]$ is left invariant.

<u>Proof</u> ~~Clearly~~ $\cancel{[(L_g)_* X, (L_g)_* Y] = [X, Y]}$, ~~so it suffices to show~~ $\cancel{[(L_g)_* X, (L_g)_* Y] = (L_g)_* [X, Y]}$.

~~Let~~ $\cancel{F : G \to \mathbb{R}}$ ~~be smooth and~~ $\cancel{h \in G}$. ~~Then~~

$$\cancel{((L_g)_*[X, Y])(F)(h) = \left((dL_g)_{g^{-1}h} [X, Y]_{g^{-1}h}\right)(F)(h)}$$

$$\cancel{= [X, Y]}$$

$$\cancel{[(L_g)_* X, (L_g)_* Y]_h F = [(L_g)_* X]_h \, [(L_g)_* Y F]}$$

$$\cancel{- [(L_g)_* Y]_h \, [(L_g)_* X F]}$$

$$(L_g)_* [X, Y](F)(h) = ((L_g)_* [X, Y])_h F$$

$$= [X, Y]_h F \circ L_g = X_h(Y F \circ L_g) - Y_h(X F \circ L_g)$$

Prop 8.16 + 8.19

$$= X_h((L_g)_* Y F) - Y_h((L_g)_* X F)$$

invariance of X and Y

$$= X_h(Y F) - Y_h(X F)$$

$$= [X, Y](F)(h)$$

$\square$

Def A <u>Lie algebra</u> is a vector space (over $\mathbb{R}$) $\mathfrak{g}$ endowed with a binary operation

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$(X, Y) \mapsto [X, Y]$$

that is bilinear, antisymmetric, and satisfies the Jacobi identity. A <u>Lie subalgebra</u> of $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. A <u>Lie algebra homomorphism</u> between $\mathfrak{g}$ and $\mathfrak{g}'$ is a linear map $A : \mathfrak{g} \to \mathfrak{g}'$ such that $[AX, AY] = A[X, Y]$ for all $X, Y \in \mathfrak{g}$.

<u>Exs</u>

- $\mathfrak{X}(M)$ is a Lie algebra (infinite dim'l)
- Let $\mathrm{Lie}(G) = \{$left invariant vector fields on $G\}$. $\mathrm{Lie}(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$, called the <u>Lie algebra</u> of G.

<u>Thm 8.37</u> The map $\varepsilon : \mathrm{Lie}(G) \to T_eG$, $X \mapsto X_e$ is a vector space isomorphism.

In particular, $\dim \mathrm{Lie}(G) = \dim G < \infty$.

Proof Clearly ε linear and injective. Why surjective? Let $v \in T_e G$ and define (rough) vector field v^L on G by

$$v^L|_g = d(L_g)_e(v)$$

Easy to see v^L left invariant ~~iff~~ (chain rule for L_{gh}) and $v^L|_e = d(L_e)_e(v) = d\,Id_e(v) = v$. So we're done if we show v^L is smooth.

To this end, let $\gamma : J \to G$ be a smooth curve with $\gamma'(0) = v \in T_e G$. For any $g \in G$ and $F \in C^\infty(G)$

$$(v^L F)(g) = v^L|_g F = d(L_g)_e(v) F = v(F \circ L_g)$$

Prop 8.16

$$= \gamma'(0)(F \circ L_g)$$

$$= \frac{d}{dt}\Big|_{t=0} (F \circ L_g \circ \gamma)(t)$$

Let φ be the smooth function

$$\varphi : J \times G \to \mathbb{R}$$
$$(t, g) \mapsto F \circ L_g \circ \gamma(t) = F(g\gamma(t)).$$

Then $\dfrac{\partial \varphi}{\partial t}(0, g) = (v^L F)(g)$ also smooth. $\square$

Corollary 8.39 Every Lie group admits a left-invariant smooth global frame. In particular, every Lie group is parallelizable.

Also worth knowing

Thm 8.44 (roughly) If $F: G \to H$ is a Lie group homomorphism, then each left G-invariant vector field on G is F-related to a unique left H invariant vector field on H. In other words, get a Lie algebra homomorphism:

$$dF_e : Lie(G) = T_e G \to T_e H = Lie(H). \quad \square$$

Lecture 8.2

Reading: First two sections of Ch. 9

Practice: 9.5

Announcement: Out of town Friday. Rather have live Zoom class, or just pre-recorded?

Integral curves

Def Let V be a vector field on M. An integral curve of V is a smooth curve $\gamma: J \to M$ such that $\gamma'(t) = V_{\gamma(t)}$ for all $t \in J$.