Corollary 8.39 Every Lie group admits a left-invariant smooth global frame. In particular, every Lie group is parallelizable.

Also worth knowing

Thm 8.44 (roughly) If $F: G \to H$ is a Lie group homomorphism, then each left G-invariant vector field on G is F-related to a unique left H invariant vector field on H. In other words, get a Lie algebra homomorphism:

$$dF_e : \mathrm{Lie}(G) = T_eG \to T_eH = \mathrm{Lie}(H). \quad \square$$

Lecture 8.2

Reading: First two sections of Ch. 9

Practice: 9.5

Announcement: Out of town Friday. Rather have live Zoom class, or just pre-recorded?

Integral curves

Def Let V be a vector field on M. An integral curve of V is a smooth curve $\gamma: J \to M$ such that $\gamma'(t) = V_{\gamma(t)}$ for all $t \in J$.

Examples

- $M = \mathbb{R}^2$, $V = \frac{\partial}{\partial x}$. Every integral curve of form $\gamma(t) = (t+a, b)$ for some $(a,b) \in \mathbb{R}^2$.

- $M = \mathbb{R}^2$, $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Every integral curve looks like $(a\cos t - b\sin t, a\sin t + b\cos t)$ for some $(a,b) \in \mathbb{R}^2$.

We call them "integral" curves because they are found by solving — or *integrating* — an autonomous system of ODEs, given initial conditions and coordinates. The existence of solutions to such (see Theorem D.1(a)) immediately implies

Prop 9.2 If V a smooth v.f. on M and $p \in M$, then there exists an $\varepsilon > 0$ and a smooth curve $\gamma: (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V with $\gamma(0) = p$. $\square$

Using full strength of Theorem D.1, we can do much better, addressing ~~which~~ "how "big" ε can be taken, and how γ depends on p. Need more terminology first

e.g. $\inf_{p \in M} \sup \varepsilon = 0$?

Flows

<u>Def</u> A (smooth) <u>global flow</u> on M is a (smooth) continuous map

$$\theta : \mathbb{R} \times M \to M$$
$$(t,p) \mapsto \theta(t,p)$$

such that

- $\theta(t, \theta(s,p)) = \theta(t+s,p) \quad \forall t,s \in \mathbb{R}, \ \forall p \in M$
- $\theta(0,p) = p \quad \forall p \in M.$

(In other words, a flow on M is a continuous group action of $\mathbb{R}$ on M.)

For each $t \in \mathbb{R}$, can define a diffeo.

$$\theta_t : M \to M$$
$$p \mapsto \theta(t,p).$$

Note that $\theta_{t+s} = \theta_t \circ \theta_s$.

For each $p \in \mathbb{R}$, can define a curve

$$\theta^{(p)} : \mathbb{R} \to M$$
$$t \mapsto \theta(t,p).$$ Note that $\theta^{(p)\prime}(0) \in T_pM$

<u>Prop 9.7</u> Let $\theta : \mathbb{R} \times M \to M$ be a smooth global flow. Then

$$V_p := \theta^{(p)\prime}(0)$$

defines a smooth v.f. on M such that each curve $\theta^{(p)}$ is integral. Call V_p the <u>infinitesimal</u>

generator of θ.

Proof To show V_p smooth, suffices to show VF is smooth for any $F \in C^\infty(M)$.

$$VF(p) = V_p F = \theta^{(p)\prime}(0) F = \frac{d}{dt}\Big|_{t=0} F(\theta^{(p)}(t))$$

$$= \frac{d}{dt}\Big|_{t=0} F(\theta(t,p))$$

Since $F \circ \theta$ is smooth, VF is smooth at p.

To show $\theta^{(p)}$ is integral, ~~consider~~ need to show that for each $s \in \mathbb{R}$, $\theta^{(p)\prime}(s) = \theta^{(q)\prime}(0)$, where $q = \theta^{(p)}(s)$. Well,

$$\cancel{\theta^{(p)}(t) = \theta(t,p) = \theta_t \theta_0(p) = \theta_t(q)} =$$

$$\theta^{(q)}(t) = \theta(t,q) = \theta(t, \theta^{(p)}(s)) \overset{\text{"}\theta(t,\theta(s,p))\text{"}}{=} \theta(t+s, p)$$

$$= \theta^{(p)}(t+s)$$

In particular,

$$\theta^{(q)\prime}(0) = \frac{d}{dt}\Big|_{t=0} \theta^{(p)}(t+s) = \theta^{(p)\prime}(s). \qquad \square$$

Examples

Each of the v.f.'s on $\mathbb{R}^2$ given earlier is the infinitesimal generator of a global flow.

- $\tau : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$
  $(t, x, y) \mapsto (x+t, y)$

- $\theta : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$
  $(t, x, y) \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$

Unfortunately, not every v.f. is the infinitesimal generator of a smooth global flow. Those that are are called __complete__ v.f.'s, and always have integral curves defined for all $t \in \mathbb{R}$ (i.e. can take $\varepsilon = +\infty$ in Prop 9.2)

Nonexamples

- Silly: let $M = \mathbb{R}^2 - \{0\}$ and $V = \frac{\partial}{\partial x}$ as before.

- Less silly: $M = \mathbb{R}$, $V = x^2 \frac{d}{dx}$. Any ~~unique~~ integral curve γ with $\gamma(0) = 1$ is of form
  $\gamma(t) = \frac{1}{1-t}$ for $t \in J$ where $J \subseteq (-\infty, 1)$.

We can fix these issues by allowing for more general definition of flow.

Def A *flow domain* for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ such that for every $p \in M$, the set

$$\mathcal{D}^{(p)} := \{t \in \mathbb{R} \mid (t,p) \in \mathcal{D}\} \subseteq \mathbb{R}$$

is an open interval containing 0. A (smooth) *flow* on M is a continuous (smooth) map $\theta : \mathcal{D} \to M$ such that

- $\theta(0,p) = p \quad \forall p \in \mathbb{R}$, and
- for all $s \in \mathcal{D}^{(p)}$, $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s+t \in \mathcal{D}^{(p)}$, have
$$\theta(t, \theta(s,p)) = \theta(t+s, p).$$

Exactly as before, can define for each ~~$p \in M$ and~~ $(t,p) \in \mathcal{D}$ ~~integral curve~~

$$\theta_t(p) = \theta^{(p)}(t) = \theta(t,p).$$

and have

Prop 9.11 If $\theta : \mathcal{D} \to M$ a smooth flow, then

$$V_p := \theta^{(p)\prime}(0)$$

is a smooth v.f. on M called the *infinitesimal generator* of θ. Each $\theta^{(p)}$ is an integral curve of V.

Def An $\overset{\text{smooth}}{\text{flow}}$ $\theta : \mathcal{D} \to M$ is <u>maximal</u> if there does not exist a larger flow domain $\tilde{\mathcal{D}} \supseteq \mathcal{D}$ and a $\overset{\text{smooth}}{\text{flow}}$ on $\tilde{\mathcal{D}}$ extending θ.

<u>**Thm 9.12**</u> (Fundamental theorem on flows)

If V is a smooth v.f. on M, then there exists a <u>unique</u> maximal smooth flow on M whose infinitesimal generator is V. We call this the <u>flow generated</u> by V. See book for more details, e.g. (a) for each $p \in M$, $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is <u>unique</u> maximal integral curve w/ $\theta^{(p)}(0) = p$.

<u>Ex</u> What is the ~~maximal~~ domain of the flow on $\mathbb{R} - \{0\}$ generated by

$$V|_x = \frac{d}{dx} ?$$

Rather than prove this (proof uses full strength of Theorem D.1), let's note some situations that guarantee completeness: Note that V is complete iff $\mathcal{D} = \mathbb{R} \times M$.

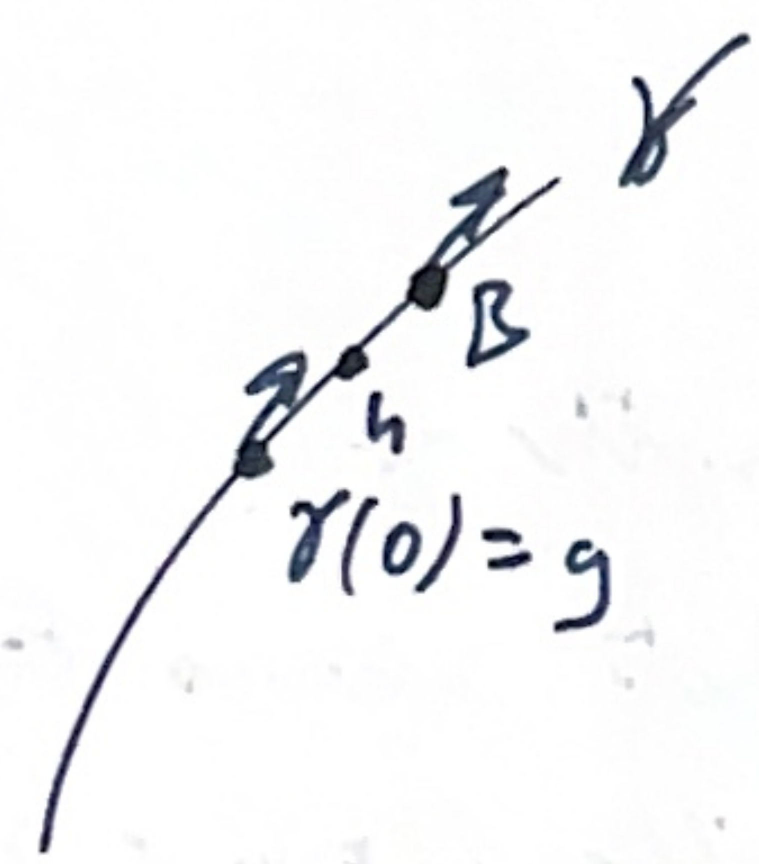
<u>**Corollary 9.17**</u> On a compact smooth manifold, every flow is complete.

<u>**Theorem 9.18**</u> Every left-invariant v.f. on a Lie group is complete.

Proof Let $X \in \mathrm{Lie}(G)$ and let $\theta : \mathcal{D} \to G$ be the flow generated by X. It suffices to show that ~~if $\theta^{(g)} : J \to$~~ every maximal integral curve ~~can be taken~~ has domain $\mathbb{R}$. Suppose, by contradiction $\theta^{(g)}$ is an maximal integral curve starting at g with domain J where $\sup J = B < \infty$. ~~Now let us assume~~ ~~$B + \varepsilon \in J$. Define $\gamma := B + \varepsilon$~~ Let $h = \theta^{(g)}(B/2)$.

~~$\gamma(t) := \begin{cases} \theta^{(g)}(t), & t \in J \\ \theta^{(h)}, & \end{cases}$~~

By left invariance of X, $\theta^{(h)}$ has maximal domain J, too, (i.e., since $\theta^{(h)} = L_h \circ \theta^{(g)}$). But then



$$\gamma(t) := \begin{cases} \theta^{(g)}(t) & \text{for } t \in J \\ \theta^{(h)}\left(t - \frac{B}{2}\right) & \text{for } t \in J + B/2. \end{cases}$$

is also an integral curve starting at g with a larger domain of definition. $\square$