Lecture 9.1

Reading: Sections on Lie derivatives & commuting v. Fs

Practice: 9.37, 9.40

Proof of Lemma 9.36

Let θ be flow of V and, in coordinates, let $\theta = (\theta^1(t,x), \dots, \theta^n(t,x))$. Then

$$d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)} M \to T_x M$$

is the matrix

$$\left(\frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) \right).$$

Thus

$$d(\theta_{-t})_{\theta_t(x)}(W_{\theta_t(x)}) = \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) W^j(\theta(t,x)) \frac{\partial}{\partial x^i}\Big|_x$$

Clearly smooth. Hence, so is $\mathcal{L}_V W = \frac{\partial}{\partial t}\Big|_{t=0}$ ". $\square$

Proof of Theorem 9.38

Suffices to show that $\mathcal{L}_V W_p = [V, W]_p$ $\forall p \in M$.

If $p \in M - \operatorname{supp} V$, easy to argue both sides 0.

If $V_p \neq 0$, choose coordinates (u^i) so that $V = \frac{\partial}{\partial u^1}$ in them. (How?) In such coordinates, have $\theta_t(u) = (u^1 + t, u^2, \ldots, u^n)$. For any pertinent t, have $d(\theta_{-t})_{\theta_t(x)} = \mathrm{Id}$. So, on one hand, have

$$d(\theta_{-t})_{\theta(t)}\left(W_{\theta_t(u)}\right) = d(\theta_{-t})_{\theta(t)}\left(W^j(u^1+t, \ldots, u^n) \frac{\partial}{\partial u^j}\Big|_{\theta_t(u)}\right)$$

$$= W^j(u^1+t, \ldots, u^n) \frac{\partial}{\partial u^j}\Big|_u .$$

$$\Rightarrow \mathcal{L}_V W = \frac{\partial W^j}{\partial u^1}(u^1, \ldots, u^n) \frac{\partial}{\partial u^j}\Big|_u .$$

On other hand,

$$[V, W] = \left(V^i \frac{\partial W^j}{\partial u^i} - W^i \frac{\partial V^j}{\partial u^i}\right) \frac{\partial}{\partial u^j}\Big|_u$$

$$= \frac{\partial W^j}{\partial u^1} \frac{\partial}{\partial u^j} = \mathcal{L}_V W.$$

Finally, for $p \in \operatorname{supp} V$, argue by continuity using that we've proved statement already on dense subset. $\square$

Commuting vector fields

Def $V, W \in \mathfrak{X}(M)$ <u>commute</u> if $[V, W] = 0$.

Def W is <u>invariant under V</u> if

$$d(\theta_t)(W_p) = W_{\theta_t(p)}$$

for all (t, p) in the domain of the flow θ generated by V.

<u>Thm 9.42</u> TFAE

(a) V, W commute

(b) W is invariant under V

(c) V " " " W.

<u>Proof</u> Clearly (c) $\Leftrightarrow$ (a) iff (b) $\Leftrightarrow$ (a), since $[V, W] = 0 \Leftrightarrow [W, V] = 0$.

(b) $\Rightarrow$ (a): easy.

(a) $\Rightarrow$ (b): we'll use the following:

Prop 9.41. For any (t_0, p) in domain of θ,

$$\left.\frac{d}{dt}\right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = d(\theta_{-t_0})\left((\mathcal{L}_V W)_{\theta_{t_0}(p)}\right).$$

Let $X(t) = d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right)$ (for $t \in \mathcal{D}^{(p)}$).
By the proposition, $X'(t) = 0$. Since $X(0) = W_p$,
this means $X(t) = W_p$ $\forall t$. Thus

$$d(\theta_t)_p X(t) = d(\theta_t)_p W_p$$

$$\Rightarrow \quad W_{\theta_t(p)} = d(\theta_t)_p W_p. \qquad \square$$

Proof of 9.41

Let $p \in M$ and $\mathcal{D}^{(p)} \subseteq \mathbb{R}$ the domain of the integral curve $\theta^{(p)}$. Consider

$$X : \mathcal{D}^{(p)} \to T_p M$$
$$t \mapsto d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right).$$

(Smooth by proof of 9.36 from earlier)

Let $t = t_0 + s$. Then

$$X'(t_0) = \frac{d}{ds}\Big|_{s=0} X(t_0+s) = \frac{d}{ds}\Big|_{s=0} d(\theta_{-t_0-s})(W_{\theta_{s+t_0}(p)})$$

$$= \frac{d}{ds}\Big|_{s=0} d(\theta_{-t_0}) \circ d(\theta_{-s})(W_{\theta_{s+t_0}(p)})$$

$$= d(\theta_{-t_0})\left(\frac{d}{ds}\Big|_{s=0} d(\theta_{-s})(W_{\theta_{s+t_0}(p)})\right)$$

$$= d(\theta_{-t_0})(\mathcal{L}_V W_{\theta_{t_0}(p)}).$$ $\square$

<u>Corollary 9.43</u> Every smooth vector field is invariant under its own flow. $\square$

<u>Def</u> If ~~...~~ θ, ψ are <u>global</u> flows on M, we say θ, ψ <u>commute</u> if $\theta_t \circ \psi_s = \psi_s \circ \theta_t$ for all $t, s \in \mathbb{R}$.

<u>Def</u> If θ, ψ are (local) flows on M, we say they <u>commute</u> if the following holds for all $p \in M$: whenever J, K open intervals containing 0 such that at least one of

$\theta_t \circ \psi_s(p)$ or $\psi_s \circ \theta_t(p)$ is defined, then both are, and they are equal.

<u>Theorem 9.44</u> Smooth vector fields commute iff their flows commute.

<u>Proof</u> For convenience, will assume our v.F's complete. Let θ be flow of V, ψ flow of W.

($\Longrightarrow$) If V, W commute, then we know V invariant under ψ. For any s, consider

$$\gamma : \mathbb{R} \to M$$
$$t \mapsto \psi_s \circ \theta_t(p) = \psi_s(\theta^{(p)}(t)).$$

Note $\gamma(0) = \psi_s(p)$ and

$$\gamma'(t) = \frac{d}{ds}\left(\psi_s(\theta^{(p)}(t))\right) \underset{\text{chain rule}}{=} d(\psi_s)(\theta^{(p)\prime}(t))$$

$$= d(\psi_s)(V_{\theta^{(p)}(t)}) = V_{\gamma(t)}.$$

V invariant under ψ.

Thus, γ is an integral curve of V starting at p. By uniqueness, $\gamma(t) = \theta^{\psi_s(p)}(t) = \theta_t(\psi_s(p))$.

($\Leftarrow$) The hypothesis equivalent to

$$\gamma^{\theta_t(p)}(s) = \theta_t(\gamma^{(p)}(s))$$

for all $s, t \in \mathbb{R}$, $p \in M$. ~~Taking derivative w.r.t. s,~~ get

$$\Rightarrow \quad \left.\frac{d}{ds}\right|_{s=0} \gamma^{\theta_t(p)}(s) = \left.\frac{d}{ds}\right|_{s=0} \theta_t(\gamma^{(p)}(s))$$

$$\Leftrightarrow \quad W_{\theta_t(p)} = d(\theta_t)_p(W_p)$$

$$\Rightarrow \quad d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = d(\theta_{-t})_{\theta_t(p)}\, d(\theta_t)_p(W_p)$$

$$= W_p.$$

$$\Rightarrow \quad \underbrace{\left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})}_{\mathcal{L}_V W} = \left.\frac{d}{dt}\right|_{t=0} W_p = 0.$$

$\square$