Lecture 9.2

Reading: First two sections of Ch. 10

Practice: 10.1, 10.9, 10.11, 10.14

Homework: 10-1, 10-10

Vector bundles

Basic idea: Möbius strip vs. TS^1.

Def A <u>(smooth) (real) vector bundle of rank k</u> over a (smooth) manifold M is a (smooth) manifold E together with a surjective (smooth) map $\pi : E \to M$ such that:

(i) $\forall p \in M$, the fiber $E_p := \pi^{-1}(p)$ is a (real) vector space.

(ii) $\forall p \in M$, exists a (smooth) <u>local trivialization</u>, meaning an open neighborhood $p \in U \subseteq M$ and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ for which $\pi_1 \circ \Phi = \pi|_U$ (here $\pi_1 : U \times \mathbb{R}^k \to U$ natural projection) and $\Phi|_{E_q} : E_q = \pi^{-1}(q) \to \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is a vector space isomorphism.

Call E the <u>total space</u>, π the <u>projection</u> and M the <u>base</u>.

A vector bundle $\pi : E \to M$ is <u>trivial</u> if exists a <u>global trivialization</u>, which is a local trivialization with over domain all of M.

<u>Example</u> $E = M \times \mathbb{R}^k$, $\pi : E \to M$, $(p,v) \mapsto p$ is called a <u>product bundle</u>.

It is trivial.

<u>Homework</u>: Möbius bundle is non-trivial.

Hint: intermediate value theorem $\Rightarrow$ every section has a zero.

<u>Example</u> TM is a rank $n = \dim M$ vector bundle over M. Prove it! (Promote a smooth chart to a local trivialization: $(U, (x^i)) \rightsquigarrow \Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$, $v^i \frac{\partial}{\partial x^i}\Big|_p \mapsto (p, (v^1, \dots, v^n))$.)

Just like the structure of a smooth manifold is captured by its transition functions, similar thing holds for vector bundles.

<u>Lemma 10.5</u> Given $\pi : E \to M$ vector bundle and local trivializations $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$, the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form $\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$ where $\tau : U \cap V \to GL(k, \mathbb{R})$ is smooth.

Call τ the <u>transition function</u> between the local trivializations Φ and Ψ.

Proof

$$\begin{tikzcd}
(U\cap V)\times\mathbb{R}^k \arrow[rd, "\pi_1"'] \arrow[r, bend right, "\Psi^{-1}"'] & \pi^{-1}(U\cap V) \arrow[l, "\Psi"'] \arrow[r, "\Phi"] \arrow[d, "\pi"] & (U\cap V)\times\mathbb{R}^k \arrow[ld, "\pi_1"] \\
 & U\cap V &
\end{tikzcd}$$

$\pi_1 = \pi_1 \circ \Phi \circ \Psi^{-1}$.

$\Rightarrow$ $\Phi\circ\Psi^{-1}(p,v) = (p, \sigma(p,v))$ for some $\sigma : (U\cap V)\times\mathbb{R}^k \to \mathbb{R}^k$ smooth. For each p, $\Phi\circ\Psi^{-1}(p,-) : \{p\}\times\mathbb{R}^k \to \{p\}\times\mathbb{R}^k$ is an invertible linear map, hence can write $\Phi\circ\Psi^{-1}(p,-) = \tau(p)-$ for $\tau : U\cap V \to GL(k,\mathbb{R})$.

Conclude by arguing that τ is smooth. For this, suffices to show that the matrix entries for $\tau(p)$ depend smoothly on p. The (i,j) entry of $\tau(-)$ is controlled by

$$(U\cap V)\times\mathbb{R} \xrightarrow{i^j} (U\cap V)\times\mathbb{R}^k \xrightarrow{\Phi\circ\Psi^{-1}} (U\cap V)\times\mathbb{R}^k \xrightarrow{\pi^i} (U\cap V)\times\mathbb{R}^k$$

$$(p,r) \mapsto (p, re_j) \qquad (p,v) \mapsto (p, v^i).$$

That is, $\tau(-)_{i,j} = \pi_i \circ \Psi^{-1} \circ \Phi \circ i^j(-,1)$. Smooth! $\square$

Just like charts can be used to define a smooth manifold <u>completely</u> (including its <u>topology</u>)

have something similar for bundles. See Lemma 10.6 (Vector bundle chart lemma).

Can use this to give new examples.

<u>Def</u> If $E' \to M$ and $E'' \to M$ are smooth vector bundles of rank k', k'' (resp.), can build a rank $k = k' + k''$ bundle by taking "fiber-wise direct sum". In particular, if

$$\tau' : U \cap V \to GL(k', \mathbb{R}) \quad \text{and} \quad \tau'' : U \cap V \to GL(k'', \mathbb{R})$$

are transition functions, then the <u>Whitney sum</u> is the vector bundle with transition functions

$$\tau : U \cap V \to GL(k' + k'', \mathbb{R})$$
$$p \mapsto \left(\begin{array}{c|c} \tau'(p) & 0 \\ \hline 0 & \tau''(p) \end{array}\right) = \tau'(p) \oplus \tau''(p).$$

<u>Ex</u> If (U, φ), (V, ψ) are stereographic coordinates on S^n, then can define a line bundle on S^n by declaring the transition function $\tau : U \cap V \to GL(1, \mathbb{R}) = \mathbb{R}^*$. When $n = 1$, get
$$p \mapsto -1$$
Möbius.

Sections

Def A (smooth) local section of bundle $\pi: E \to M$ is an open $U \subseteq M$ and a (smooth) $\sigma: U \to E$ such that $\pi \circ \sigma = Id|_U$. If $U = M$, call this a global section.

Def The zero section of $\pi: E \to M$ is the global section
$$\sigma: M \to E$$
$$p \mapsto 0_p \in E_p.$$

Def $\Gamma(\pi: E \to M) := \{\text{all smooth global sections } \sigma: M \to E\}$. Infinite dimensional vector space. Module over $C^\infty(M)$.

Exs - $\Gamma(\pi: TM \to M) = \mathfrak{X}(M)$.

- $\Gamma(\pi: M \times \mathbb{R} \to \mathbb{R}) = C^\infty(M)$.

  trivial

- $\Gamma(M \times \mathbb{R}^k \to M) = C^\infty(M, \mathbb{R}^k)$.

Def A <u>local frame</u> for $\pi : E \to M$ (rank k!) is a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ over some $U \subseteq M$ open such that $(\sigma_1(p), \dots, \sigma_k(p))$ is linearly independent in E_p (hence, spans) for all $p \in U$. If $U = M$, called a <u>global section</u>.

<u>Corollary 10.20</u> A (smooth) vector bundle is (smoothly) trivial iff it has a (smooth) global frame. In particular, TM trivializable iff M parallelizable.

<u>Proof</u> Second statement is special case of first.

($\Longrightarrow$): Easy construction.

($\Longleftarrow$): Let $(\sigma_1, \dots, \sigma_k)$ be a global frame for E.

Define a map

$$F : M \times \mathbb{R}^k \to E$$
$$(p, (v^1, \dots, v^k)) \mapsto v^i \sigma_i(p) \in E_p.$$

Clearly linear isomorphism for each fixed p. Also clearly bijective. Thus, can finish if we can show it is a local diffeo. To this end, let's consider what F looks like "in coordinates," i.e. on a local trivialization (U, Φ):

$$\begin{tikzcd}
\cancel{M} U \times \mathbb{R}^k \arrow[r, "F|_{U\times\mathbb{R}^k}"] \arrow[dr, "\pi_1"'] & \pi^{-1}(U) \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\
& U \arrow[u, dashed, bend left, "\sigma_i"] &
\end{tikzcd}$$

For each $\sigma_i : U \to \pi^{-1}(U)$, postcomposing with Φ and taking coordinate components in $\mathbb{R}^k$ leads to $\sigma_i^1, \dots, \sigma_i^k : U \to \mathbb{R}$ such that

$$\Phi \circ \sigma_i(p) = (p, (\sigma_i^1(p), \dots, \sigma_i^k(p)))$$

~~Since Φ and $F|_{U\times\mathbb{R}^k}$ are both linear isomorphisms on fibers,~~ By def'n of F,

$$\Phi \circ F|_{U\times\mathbb{R}^k}(p, v) = \Phi \circ F|_{U\times\mathbb{R}^k}(p, (v^1, \dots, v^k))$$

$$= (p, (v^i \sigma_i^1(p), \dots, v^i \sigma_i^k(p))),$$

which is smooth.

It is straightforward to verify that $(\Phi \circ F|_{U\times\mathbb{R}^k})^{-1}$ is $(p, w) \mapsto (p, A(p) w)$, where $A(p) = (\sigma_i^j(p))^{-1}$.

Since inversion is smooth on $GL(k, \mathbb{R})$, conclude $(\Phi \circ F|_{U\times\mathbb{R}^k})^{-1}$ smooth. $\square$