# Lecture 9.3

**Reading**: Second half of Ch. 10

**Practice**: 10.27, 10.31.

## Bundle maps

**Def** If $\pi : E \to M$, $\pi' : E' \to M'$ two smooth vector bundles, a smooth map $F : E \to E'$ is a <u>bundle homomorphism</u> if exists a map $f : M \to M'$ such that $\pi' \circ F = f \circ \pi$ and $F|_{E_p} : E_p \to E'_{f(p)}$ linear.

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow[f]{} & M' \end{array}$$

**Intuition**: a smoothly varying family of linear maps (between smoothly varying families of vector spaces).

If a bundle homomorphism is a diffeomorphism we call it a <u>bundle isomorphism</u>.

We say $F$ <u>covers</u> $f$.

**Prop 10.25** If $F$ a bundle homomorphism, then there ~~map~~ is a unique map it covers.

**Proof** $F$ must be

$$f(p) = \pi'(F(v)) \quad \text{where } v \in \pi^{-1}(p) = E_p. \quad \square$$

**Example** If $F: M \to N$ smooth, then $dF: TM \to TN$ a bundle homomorphism covering $F$.

**Note:** If $\pi: E \to M$ and $\pi': E' \to M$ have same base, we typically only consider bundle homomorphisms "over $M$," meaning they cover $id_M$ (and not some other $F: M \to M$).

**Example** Let $f \in C^\infty(M)$, get a smooth map

$$F: TM \to TM$$
$$v_p \mapsto f(p)v \quad \text{where } v \in T_pM.$$

# Subbundles

**Def** If $\pi : E \to M$ is a (smooth) vector bundle, then a <u>subbundle</u> of $\pi$ is an embedded submanifold $D \subseteq E$ such that $D \cap E_p$ is a vector subspace of $E_p$ for all $p \in M$, and $\pi|_D : D \to M$ is a smooth vector bundle.

**Ex** Let $V \in \mathfrak{X}(M)$ be nowhere vanishing. Then $D := \{(p,v) \in TM \mid (p,v) \in \operatorname{span} V_p\}$ is a smooth rank 1 bundle that is a subbundle of $TM$. Note $D$ is trivial!

**Ex** If $S \subseteq M$ immersed, then $TS$ is a subbundle of $TM$.

**Def** A bundle homomorphism $F : E \to E'$ has <u>constant rank</u> if $F|_{E_p}$ has constant rank $\forall p \in M$.

Ex $F: M \to N$ has constant rank if $dF: TM \to TN$ does.

<u>Theorem 10.34</u> Let $E, E'$ be vector bundles over $M$ (same base!), and $F: E \to E'$ be a bundle map over $M$. Define

$$\operatorname{Ker} F := \bigcup_{p \in M} \ker F|_{E_p}, \quad \operatorname{Im} F := \bigcup_{p \in M} \operatorname{Im} F|_{E_p}.$$

Then $\operatorname{Ker} F$ and $\operatorname{Im} F$ are smooth subbundles if and only if $F$ is constant rank.

<u>Proof</u>: ($\Longrightarrow$): easy.

($\Longleftarrow$): Use the "local frame criterion for subbundles": suppose $\forall p \in M$, given subspaces $D_p \subseteq E_p$ of dimension $k$. Then $D := \bigcup_{p \in M} D_p$ a subbundle iff exist local sections $\sigma_1, \dots, \sigma_k : U \to E$ with $\sigma_i(U) \subseteq D$ and $(\sigma_1(p), \dots, \sigma_k(p))$ a basis of $D_p$.

Because $F$ has constant rank (call it $r$), if we look in local trivializations of $E, E'$

$$\begin{tikzcd}
U \times \mathbb{R}^k & \pi^{-1}(U) \arrow[l, "\Phi"'] \arrow[rr, "F"] \arrow[dr, "\pi"] & & (\pi')^{-1}(U) \arrow[r, "\Psi"] \arrow[dl, "\pi'"'] & U \times \mathbb{R}^{k'} \\
 & & U & &
\end{tikzcd}$$

we see $\Psi \circ F \circ \Phi^{-1}(p,v) = (p, A(p)v)$, where $A(p)$ is ~~a rank~~ $k' \times k$ matrix with rank $r$ ~~that depends~~ independent of $p$. If $(\sigma_1, \dots, \sigma_k)$ is the local frame of $E$ determined by $\Phi$, then (possibly after permuting the coordinates of $\mathbb{R}^k$), $(F \circ \sigma_1, \dots, F \circ \sigma_r)$ will be a local frame for $\operatorname{Im} F$ satisfying the local frame criterion. Thus, $\operatorname{Im} F$ is a smooth subbundle of $E'$ whenever $F$ has constant rank.

To show $\operatorname{Ker} F$ is a smooth subbundle, use the fact that $\operatorname{Ker} A(p) = (\operatorname{Im} A(p)^T)^\perp$ and argue similarly using local sections of $E'$ (and the fact that $F$ constant rank $\Rightarrow$ $A(p)^T$ constant rank). $\square$