CS 593/MA 592 - Intro to Quantum Computation
Homework 1

Due Monday, January 15 at 8pm (upload to Brightspace)

There are a lot of exercises on the homework this week, but the vast majority of them are drills, which you are hopefully doing while you read the textbook anyway. If you submit your homework with a partner, then I strongly encourage you to make sure you both do some of the parts of all of the different exercises.

1. Get comfortable with bra-ket notation by doing the following exercises out of Section 2.1 of Nielsen and Chuang: 2.2, 2.7, 2.10, 2.20, 2.26

2. Get even more comfortable with bra-ket notation by doing the following two exercises.

   (a) Let $\mathcal{H}$ be a finite dimensional Hilbert space. We define the *dual* Hilbert space $\mathcal{H}^*$ to be the set of all linear transformations $\rho : \mathcal{H} \to \mathbb{C}$, that is:

   $$\mathcal{H}^* := \{\rho : \mathcal{H} \to \mathbb{C} \mid \rho \text{ is a linear function}\}.$$

   Recall that if $|\psi\rangle \in \mathcal{H}$, then $\langle\psi|$ is the linear transformtation

   $$\begin{aligned} \langle\psi| : \mathcal{H} &\to \mathbb{C} \\ |\phi\rangle &\mapsto \langle\psi \mid \phi\rangle \end{aligned}$$

   where $\langle\psi \mid \phi\rangle$ is the inner product.

   i. Show that $\mathcal{H}^*$ is a vector space with the same dimension as $\mathcal{H}$.

   ii. Show that the function

   $$\begin{aligned} F : \mathcal{H} &\to \mathcal{H}^* \\ |\psi\rangle &\to \langle\psi| \end{aligned}$$

   is a bijection. Warning: it is NOT linear (it is *anti-linear* or *conjugate-linear*), so you more-or-less need to show injectivity and surjectivity directly. For surjectivity, use the previous problem part (in particular, the fact that $\mathcal{H}$ is finite dimensional).

   In fact, we won't do this, but it's even possible to define an inner product structure on $\mathcal{H}^*$, and the map $|\psi\rangle \mapsto \langle\psi|$ becomes an anti-linear isometry. The moral of the story is that Hilbert spaces are ALMOST isometrically isomorphic to their dual spaces—the only finnicky thing is that the "isomorphism" is not linear, it's anti-linear! This is called the Riesz representation theorem. It's true more generally, *i.e.* for infinite dimensional Hilbert spaces too.

   (b) Let $\mathcal{B}(\mathcal{H})$ be the set of all linear transformations $A : \mathcal{H} \to \mathcal{H}$.

   i. Show that $\mathcal{B}(\mathcal{H})$ is a vector space. What is its dimension?

   ii. Show that the map

   $$\begin{aligned} \mathcal{H} \otimes \mathcal{H}^* &\to \mathcal{B}(H) \\ |\phi\rangle \otimes \langle\psi| &\mapsto |\phi\rangle\langle\psi| \end{aligned}$$

   is a vector space isomorphism.

3. In this set of exercises, we dig into the spectral theorem/diagonalization in more detail.

   (a) Prove the corollaries of the spectral theorem (Theorem 2.1 in Nielsen and Chuang) that I stated on Tuesday by doing exercises 2.17 and 2.18. [Hint: being "corollaries" of course means that you should use the spectral theorem in your proof.]

   (b) Differing slightly from the book (see page 70), in this problem let us *define* an orthogonal projector to be a linear operator $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$ and $P^* = P$. Show that P is an orthogonal projector if and only if P is unitarily diagonalizable, with all eigenvalues equal to either 0 or 1.

   (c) Do exercises 2.29-2.32. [Hint: you can use the previous parts of this exercise, or, closely related, Exercise 2.28 (which you need not prove for this problem).]

4. Get familiar with commutators by doing exercises 2.42, 2.44, 2.45, 2.46 and 2.47.