CS 593/MA 595 - Intro to Quantum Computation
Theoretical Homework 9

Due Wednesday, December 3 at 11:59PM (upload to Brightspace)

1. Show that the 1-Local Hamiltonian Problem is in P.

2. Consider the simulation of a time-dependent Hamiltonian $H(t)$, which is the solution $U(0,T)$ (from time $t=0$ to T) to the Schrödinger equation (for unitary) $i\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H(t)U(t)$. Assume that $H(t)$ is (first-order) differentiable and bounded by $M = \max_{0\le t\le T}||H(t)||$ and $D = \max_{0\le t\le T}||H'(t)||$. We can approximate it with piece-wise time-independent Hamiltonian at N equidistant time stamps $t_j = (j-1)T/N$, each evolving for T/N (with algorithms introduced in classes), and bound the errors.

   (a) Let $V_j(t) = e^{iH(t_j)t}U(t_j, t_j + t)$ for $0 \le t < \frac{T}{N}$. Show that
   $$\frac{\mathrm{d}}{\mathrm{d}t}V_j(t) = i\left(e^{iH(t_j)t}\left(H(t_j) - H(t)\right)U(t_j, t_j+t)\right).$$

   (b) Show:
   $$\left\| U(t_j, t_{j+1}) - e^{-iH(t_j)\frac{T}{N}} \right\| \le \frac{DT^2}{2N^2}.$$
   [Hint: rewrite the left hand side to $e^{-iH(t_j)\frac{T}{N}}(V_j(t) - I)$, and take derivative.]

   (c) Prove that
   $$\left\| U(0,T) - \prod_{j=N}^{1} e^{-iH(t_j)\frac{T}{N}} \right\| \le \frac{DT^2}{2N}.$$

3. Prove: if there is a fast-forwarding quantum simulation, then $\mathsf{EXP} = \mathsf{BQP}$. A more rigorous context:

   - We assume that we have a quantum algorithm that can simulate arbitrary time-independent Hamiltonian H for time t, succinctly described by a classical string x of length n. To simulate within error ϵ, the quantum algorithm has gate complexity $O(t^{1-\delta}\mathrm{poly}(\log t, \epsilon^{-1}, n))$ for a small constant $\delta > 0$.
   - By "succinctly described by x of length n", we assume that H is on $O(\mathrm{poly}(n))$ qubits, $\|H\| = O(\mathrm{poly}(n))$, and there is an classical polynomial-time algorithm computing $t = O(2^{\mathrm{poly}(n)})$ and H_{jk} given x, j, k.

     Example: bounded row-sparse Hamiltonian and bounded Ising models can be succinctly described.
   - EXP is the problem class that can be decided by deterministic Turing machines that terminates in exponential time (no assumptions on space). It is also known that $\mathsf{BQP} \subseteq \mathsf{QEXP} = \mathsf{EXP}$.

   An outline of the proof with sub-problems is provided in the next page. You are recommended to think about the high-level idea (which is easy and fun) before reading the next page (which contains many details).

Prove the following arguments:

(a) For any unitary U, let

$$H = -\begin{bmatrix} 0 & U^\dagger \\ U & 0 \end{bmatrix} = -\left|1\right\rangle\left\langle 0\right| \otimes U - \left|0\right\rangle\left\langle 1\right| \otimes U^\dagger,$$

then

$$e^{-iH\frac{\pi}{2}}\left|0\right\rangle\left|\psi\right\rangle = i\left|1\right\rangle(U\left|\psi\right\rangle).$$

(b) (Feynman-Kitaev) Given a quantum circuit $U = U_L \cdots U_1$ succinctly described by a classical string x of length n, there is a time-independent and succinctly described Hamiltonian H on $O(\mathrm{poly}(n))$ qubits such that $I \otimes U = \exp(-iHL)$ up to a global phase.

Here, the circuit U being succinctly described by x means that U acts on $O(\mathrm{poly}(n))$ qubits, and there is a classical polynomial-time algorithm that can compute $L = O(2^{\mathrm{poly}(n)})$ and specify U_j, an elementary 1- or 2-qubit quantum gate, given x and j.

[Hint: Consider introducing an ancillary register t whose value ranges from 0 to L, and Hamiltonian of a generalized form from part (a).]

(c) With the fast-forwarding quantum algorithm simulating a succinctly described evolution with gate complexity $O(t^{1-\delta}\mathrm{poly}(\log t, \epsilon^{-1}, n))$, there is a faster quantum algorithm simulating a succinctly described evolution with $O(t^{(1-\delta)^2}\mathrm{poly}(\log t, \epsilon^{-1}, n))$ gates.

(d) With the fast-forwarding quantum algorithm, there is a quantum algorithm simulating a succinctly described evolution with $O(\mathrm{poly}(\log t, \epsilon^{-1}, n))$ gates.

[Hint: bootstrap part (c) for $O(\log\log t)$ times]

(e) **(Not required to prove)** (Chandra–Kozen–Stockmeyer) $\mathsf{APSPACE} = \mathsf{EXP}$. Hence, for any problem in EXP and input length n, there is a classical circuit with $O(\mathrm{poly}(n))$ width and $O(2^{\mathrm{poly}(n)})$ depth to determine it.

(f) **(Not required to prove)** For a classical circuit with width w and depth l, there exists a quantum circuit with $O(wl)$ gates and $O(w\log(l))$ qubits simulating the classical circuit.

(g) With the fast-forwarding quantum algorithm, any problem in EXP can be solved with a polynomial time quantum algorithm, which is $\mathsf{EXP} \subseteq \mathsf{BQP}$.