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Global behavior of a multi-group SIS epidemic model with age structure

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Abstract

We study global dynamics of a system of partial differential equations. The system is motivated by modelling the transmission dynamics of infectious diseases in a population with multiple groups and age-dependent transition rates. Existence and uniqueness of a positive (endemic) equilibrium are established under the quasi-irreducibility assumption, which is weaker than irreducibility, on the function representing the force of infection. We give a classification of initial values from which corresponding solutions converge to either the disease-free or the endemic equilibrium. The stability of each equilibrium is linked to the dominant eigenvalue $s(\mathbf{A})$, where $\mathbf{A}$ is the infinitesimal generator of a "quasi-irreducible" semigroup generated by the model equations. In particular, we show that if $s(\mathbf{A}) < 0$ then the disease-free equilibrium is globally stable; if $s(\mathbf{A}) > 0$ then the unique endemic equilibrium is globally stable.

Keywords: Partial differential equations; Global stability; Quasi-irreducibility; Threshold conditions; Epidemic model

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1. Introduction

Many infectious diseases transmitted by bacterial agents (e.g., tuberculosis) or sexually transmitted diseases (e.g., gonorrhea) can be studied using *SIS* epidemiology models with S and I representing the susceptible and infected individuals, respectively. While ODE models are often used when the population structures (age, sex, etc.) are neglected, there are many cases in which incorporating one or more of these structures into the model may provide additional and important information which may be helpful in the understanding of the disease dynamics. The incorporation of age-dependent demographical and/or epidemiological parameters usually leads to a system of first-order partial differential equations with nonlocal boundary conditions. This paper considers an age-structured *SIS* model.

Most existing studies on *SIS* models give only local stability results for which a variety of analytical tools are available. In contrast, global studies of these models are very limited due to the lack of applicable theories. For ODE models, a complete characterization of the global dynamics was first due to the work of Lajmanovich and York [9] by employing a Liapunov function, and was later given by Smith [12] using the monotone iteration approach. The study of *SIS* models with age-structure, which are given by first-order PDE's, involves more sophisticated technical details and the global dynamical properties in general cannot follow directly from classical theory of the monotone flows unless we assume that the flows generated by models are irreducible in a Banach lattice [16, p. 306] and possess the compactness property. These assumptions in general are too restrictive to have biological applications. The global stability results for the case of a single group age-structured model were first obtained in [2–4]. The results given in these papers require that the force of infection function satisfies some separability conditions. Under this assumption they proved the uniqueness of the positive equilibrium if it exists. In the case when a positive equilibrium exists, they provided a precise partition of a positively invariant set Ω into two subsets, Ω_1 and Ω_2, for which all solutions with initial values in Ω_1 (Ω_2) converge to the positive (zero) equilibrium.

In this paper, we study a more general age-structured *SIS* model that includes multiple groups of human populations and relaxes the irreducibility and separability conditions. This brings forth two mathematical problems. First, we need to identify a general assumption that is weaker than irreducibility and separability condition but still ensures the uniqueness of the positive equilibrium as well as the global stability result. Second, since the drop of irreducibility leads to the possibility that not all nontrivial solutions will converge to the positive equilibrium, we need to give a classification of those initial values from which the solutions converge to the positive equilibrium. The paper is organized as follows. In Section 2, we describe the multi-group model and the reduced system under the assumption that the total population of each subgroup has reached its stable age distribution. Section 3 defines the so-called "quasi-irreducibility" and presents preliminaries for "quasi-irreducible" semigroups generated by a system of linear age-structure models. Our main theorems for the nonlinear model and the proofs are given in Section 4, and an example of application of our results is provided in Section 5.

2. A multiple group model with age structure

Let us consider a population consisting of n subgroups that are exposed to an infectious disease. For each group i we use $s_i(t,a)$ and $u_i(t,a)$ to denote the age-specific densities of the susceptibles and infecteds at time t and age a, respectively. Let $b_i(a)$ denote the age-specific per capita birth rate; $\mu_i(a)$ the death rate; $\gamma_i(a)$ the cure rate in group i, and let $\omega > 0$ be the maximum life span. Our model equations are:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) s_i(t,a) &= -\mu_i(a) s_i(t,a) - \Lambda_i(a, u(t,\cdot)) s_i(t,a) + \gamma_i(a) u_i(t,a), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\right) u_i(t,a) &= -\mu_i(a) u_i(t,a) + \Lambda_i(a, u(t,\cdot)) s_i(t,a) - \gamma_i(a) u_i(t,a), \end{aligned} \tag{2.1}$$

where

$$\Lambda_i(a, u(\cdot,t)) := K_i(a) u_i(a,t) + \sum_{j=1}^{n} \int_0^{\omega} K_{ij}(a,s) u_j(s,t)\, ds$$

for $u = (u_1, \ldots, u_n)$. $K_i(a)$ is the infection rate for pure intracohort interaction in group i and $K_{ij}(a,s)$ is the rate at which an infective individual of age s in group j comes into a disease transmitting contact with a susceptible individual of age a in group i. The initial and boundary conditions of the system are given by

$$\begin{aligned} s_i(t,0) &= \int_0^{\omega} b_i(a)[s_i(t,a) + (1-q_i) u_i(t,a)]\, da, \\ u_i(t,0) &= q_i \int_0^{\omega} b_i(a) u_i(t,a)\, da, \quad 0 < q_i < 1, \\ s_i(0,a) &= \psi_i(a), \\ u_i(0,a) &= \varphi_i(a), \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.2}$$

where q_i is the fraction of newborn that is infected.

The basic reproductive number of the population in group i is

$$R_i := \int_0^{\omega} b_i(a) \exp\left(-\int_0^a \mu_i(\tau)\, d\tau\right) da, \quad i = 1, 2, \ldots, n.$$

We adopt the same assumption as in [3] that the population in each group is in a stationary demographic state. That is, $R_i = 1$, for $i = 1, 2, \ldots, n$. Under this assumption, the density function, $s_i(t,a) + u_i(t,a)$, of the total population of group i satisfies

$$\lim_{t\to\infty} s_i(t,a) + u_i(t,a) = c_i \exp\left(-\int_0^a \mu_i(\tau)\, d\tau\right), \quad a \in [0, \omega],$$

where c_i is a constant. Without loss of generality we suppose that $c_i = 1$, $i = 1, 2, \ldots, n$. We further suppose that the total population density (scaled by c_i) for group i has already reached its stable distribution:

$$s_i(t,a) + u_i(t,a) \equiv p_i(a) := \exp\left(-\int_0^a \mu_i(\tau)\,d\tau\right), \quad a \geqslant 0, \quad i = 1, \ldots, n. \tag{2.3}$$

Then replacing $s(t,a)$ by $p_i(a) - u_i(t,a)$ in System (2.1) allows us to eliminate the s equation and get the following system which is equivalent to (2.1)–(2.2):

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) u_i(t,a) &= -[\mu_i(a) + \gamma_i(a)]u_i(t,a) + \Lambda_i(a, u(t,\cdot))[p_i(a) - u_i(t,a)], \\ u_i(t,0) &= \int_0^\omega \beta_i(a) u_i(t,a)\,da, \quad t > 0, \\ u_i(0,a) &= \varphi_i(a), \quad a \geqslant 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.4}$$

where $\beta_i(a) = q_i b_i(a)$. Throughout this paper we assume the following:

(H1) $\mu_i,\ \gamma_i,\ K_i \in L^\infty([0,\omega])$, $K_{ij} \in L^\infty([0,\omega]^2)$, and $K_{ij} \geqslant 0$ for $i, j = 1, \ldots, n$.
(H2) $\int_0^\omega \beta_i(a)\,da > 0$, for $i = 1, \ldots, n$.

Furthermore, we consider the phase space of the system (2.4) to be the Banach space

$$X := \{\varphi = (\varphi_i, \ldots, \varphi_n);\ \ \varphi_i \in L^1[0,\omega],\ \ i = 1, 2, \ldots, n\}$$

equipped with the norm $\|\cdot\|_X$ defined by

$$\|\varphi\|_X = \max_{1 \leqslant i \leqslant n} \left\{ \int_0^\omega |\varphi_i(a)|\,da \right\}.$$

3. Preliminaries and quasi-irreducibility

Under the assumptions (H1) and (H2), the existence and uniqueness of solution to the problem (2.4) are well established [15]. Introduce the following notations and definitions:

1. For $\varphi,\ \psi \in X$, $\phi \leqslant \psi$ if $\varphi_i(a) \leqslant \psi_i(a)$, $a \in [0,\omega]$, $i = 1, \ldots, n$.
2. For $\varphi \in X$, $\varphi \geqslant 0$ if all components of φ are nonnegative, and $\varphi \gg 0$ if all component of φ are strictly positive.
3. $X_+ = \{\varphi \in X : \varphi \geqslant 0\}$, $X_+^\Omega = \{\varphi \in X;\ \ 0 \leqslant \varphi \leqslant p\}$ where $p = (p_1, \ldots, p_n)$.
4. An operator $T : X \to X$ is said to be *positive* if $TX_+ \subseteq X_+$.

5. Let $X^* = \{\varphi_i^* = (\varphi_i^*, \ldots, \psi_n^*); \varphi_i^* \in L^\infty([0,\omega]), i = 1, 2, \ldots, n\}$ be the dual space of X, and for $\varphi^* \in X^*$ and $\varphi \in X$,

$$\langle \varphi^*, \varphi \rangle = \sum_{i=1}^{n} \int_0^\omega \varphi_i^*(a)\varphi_i(a)\, da.$$

Let $u(t, \cdot, \varphi)$ denote the solution to (2.4). Using the same arguments as in [3,4] one can verify the following:

(1) For any $\varphi \in X_+^\Omega = \{\varphi \in X; 0 \leqslant \varphi_i(a) \leqslant p_i(a),\ a \in [0,\omega], i = 1, \ldots, n\}$, $u(t, \cdot, \varphi) \in X_+^\Omega$ for all $t \geqslant 0$.
(2) The system (2.4) introduces a monotone flow. That is, if $\varphi, \psi \in X_+^\Omega$ and $\varphi \leqslant \psi$, then $u(t, \cdot, \varphi) \leqslant u(t, \cdot, \psi)$ for all $t \geqslant 0$.

Let us first consider the linear system corresponding to (2.4). Let

$$B(a,s) = \left[B_{ij}(a,s)\right]_{n\times n}, \quad \sigma(a) = \begin{bmatrix} \sigma_1(a) & & \\ & \ddots & \\ & & \sigma_n(a) \end{bmatrix}, \quad \beta(a) = \begin{bmatrix} \beta_1(a) & & \\ & \ddots & \\ & & \beta_n(a) \end{bmatrix}$$

with

$$\begin{aligned} B_{ij}(a,s) &= p_i(a)K_{ij}(a,s), \\ \sigma_i(a) &= \mu_i(a) + \gamma_i(a) - p_i(a)K_i(a), \quad i, j = 1, 2, \ldots, n. \end{aligned} \tag{3.1}$$

Then the linear system is

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) u(t,a) &= -\sigma(a)u(t,a) + \int_0^\omega B(a,s)u(t,s)\, ds, \\ u(t,0) &= \int_0^\omega \beta(a)u(t,a)\, da, \\ u(0,a) &= \varphi(a), \quad t > 0, \quad a \in [0,\omega], \quad \varphi \in X. \end{aligned} \tag{3.2}$$

It is well known (see [10,11,15]) that (3.2) generate a strongly continuous, positive semigroup $T(t)$, $t \geqslant 0$; that is, for $\varphi \in X_+$,

$$T(t)\varphi = u(t, \cdot, \varphi) \geqslant 0, \quad t \geqslant 0.$$

The dynamics of (2.4) depend largely on the behavior of the integral kernels K_{ij}, $i, j = 1, 2, \ldots, n$. Complicated kernels can generally produce complicated dynamics.

In this paper, we consider the situation in which the population is "entirely" involved in the disease transmission processes. This may be interpreted mathematically as that the system is "*quasi-irreducible*" (which may not be a standard definition in literature).

We now give the definition of quasi-irreducibility, abbreviated as q-irreducibility. Let $\mathbf{A}$ be the infinitesimal generator of $T(t)$, that is

$$[\mathbf{A}\varphi](a) = -\dot{\varphi}(a) - \sigma(a)\varphi(a) + \int_0^\omega B(a,s)\varphi(s)\,ds,$$

$$\mathcal{D}(\mathbf{A}) = \left\{\varphi \in X : \varphi \text{ is absolutely continuous}, \;\; \varphi(0) = \int_0^\omega \beta(a)\varphi(a)\,da\right\}.$$

Since an eigenfunction of $\mathbf{A}$ is in $\mathcal{D}(\mathbf{A})$, it is in $C([0,\omega],\mathbb{R}^n)$.

Definition 3.1. The positive semigroup $T(t)$, or its generator $\mathbf{A}$, is said to be *q-irreducible* if $\mathbf{A}$ has no eigenfunction in ∂C_+ where $C_+ = \{f \in C([0,\omega],\mathbb{R}^n) : f \geqslant 0\}$.

We now investigate the properties of the q-irreducible operator of $\mathbf{A}$. Let $s(\mathbf{A})$ be the spectral bound of $\mathbf{A}$, i.e.,

$$s(\mathbf{A}) = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\mathbf{A})\}.$$

Then X^* is the dual space of X. Let $\mathbf{A}^*$ be the formal adjoint operator of $\mathbf{A}$ defined as

$$[\mathbf{A}^*\varphi^*]_i(a) = \dot{\varphi}_i^*(a) - \sigma_i(a)\varphi_i^*(a) + \sum_{j=1}^n \int_0^\omega B_{ji}(s,a)\varphi_j^*(s)\,ds + \beta_i(a)\varphi_i^*(0),$$

$$\mathcal{D}(\mathbf{A}^*) = \{\varphi^* \in X^*; \;\; \dot{\varphi}^* \in X^*, \;\; \varphi^*(\omega) = 0\}.$$

We shall show that the operator $\mathbf{A}^*$ defined above is a true adjoint operator of $\mathbf{A}$. To proof this, let us first establish the following lemmas. Let

$$C_1 = \left\{x \in C^1([0,\omega] : \mathbb{R}^1); \;\; x(0) = \int_0^\omega k(a)x(a)\,da\right\}.$$

Lemma 3.2. *Suppose* $x^*, \eta, k \in L^\infty[0,\omega]$ *with* $k \geqslant 0$ *and* $\int_0^\omega k(a)\,da > 0$. *For any* $x \in C_1$, *if*

$$\int_0^\omega x^*(a)\dot{x}(a)\,da = \int_0^\omega \eta(a)x(a)\,da, \tag{3.3}$$

then x^* *is absolutely continuous and* $\dot{x}^*(a) = -\eta(a) - x^*(0)k(a)$ *for almost every* $a \in [0, \omega]$.

Proof. Since $\eta \in L^\infty[0, \omega] \subset L^1[0, \omega]$, the function $\eta^*(a) = \int_a^\omega \eta(s)\, ds$ is absolutely continuous and $\eta^*(\omega) = 0$. Hence, for $x \in C_1$, using integration by parts [7, p. 100] in (3.3),

$$\begin{aligned}\int_0^\omega x^*(a)\dot{x}(a)\, da &= \int_0^\omega \eta(a)x(a)\, da \\ &= -\int_0^\omega x(a) d_a \eta^*(a) \\ &= x(0)\eta^*(0) + \int_0^\omega \eta^*(a)\dot{x}(a)\, da.\end{aligned}$$

Let $z^* = x^* - \eta^*$. Then the equality above implies that

$$\int_0^\omega z^*(a)\dot{x}(a)\, da = x(0)\eta^*(0) \tag{3.4}$$

for any $x \in C_1$. We fix a function $y \in C^1([0, \omega])$ with $y(0) = 0$ and $y(a) > 0$ for $a \in (0, \omega]$. Then, for any function $x \in C^1([0, \omega]$ that is strictly positive on $(0, \omega]$ and $x(0) = 0$, the assumption on k implies that

$$h = -\frac{\int_0^\omega k(a)x(a)\, da}{\int_0^\omega k(a)y(a)\, da}$$

is well defined. If we let $\phi^x = x + hy$, then $\phi^x \in C^1([0, \omega])$ and $\phi^x(0) = 0$. Moreover,

$$\int_0^\omega k(a)\phi^x(a)\, dy = \int_0^\omega k(a)x(a)\, da + h\int_0^\omega k(a)y(a)\, da = 0.$$

Hence $\phi^x \in C_1$. From (3.4),

$$\int_0^\omega z^*(a)\dot{x}(a)\, da + h\int_0^\omega z^*(a)\dot{y}(a)\, da = \int_0^\omega z^*(a)\dot{\phi}^x(a)\, da = \phi^x(0)\eta^*(0) = 0$$

or

$$\int_0^\omega z^*(a)\dot{x}(a)\, da = -h\int_0^\omega z^*(a)\dot{y}(a)\, da = \frac{\int_0^\omega k(a)x(a)\, da}{\int_0^\omega k(a)y(a)\, da}\int_0^\omega z^*(a)\dot{y}(a)\, da.$$

It follows that

$$\frac{\int_0^\omega z^*(a)\dot{x}(a)\,da}{\int_0^\omega k(a)x(a)\,da} = \frac{\int_0^\omega z^*(a)\dot{y}(a)\,da}{\int_0^\omega k(a)y(a)d\,da} = c$$

for some real number c, or equivalently

$$\int_0^\omega z^*(a)\dot{x}(a)\,da = c\int_0^\omega k(a)x(a)\,da. \tag{3.5}$$

Let $k^*(a) = \int_a^\omega k(s)\,ds$ for $a \in [0, \omega]$. Using $k^*(\omega) = x(0) = 0$, (3.5), and integration by parts we get

$$\int_0^\omega z^*(a)\dot{x}(a)\,da = -c\int_0^\omega d_a k^*(a)x(a) = c\int_0^\omega k^*(a)\dot{x}(a)\,da.$$

It follows that

$$\int_0^\omega [z^*(a) - ck^*(a)]\dot{x}(a)\,da = 0. \tag{3.6}$$

Note that (3.6) holds for any continuously differentiable function x that is strictly positive on $(0, \omega]$ with $x(0) = 0$. For any strictly positive continuous function ξ defined on $[0, \omega]$, let $x(a) = \int_0^a \xi(s)\,ds$. Then $\dot{x} = \xi$, and x is strictly positive on $(0, \omega]$ with $x(0) = 0$. It follow from (3.6) that

$$\int_0^\omega [z^*(a) - ck^*(a)]\xi(a)\,da = \int_0^\omega [z^*(a) - ck^*(a)]\dot{x}(a)\,da = 0$$

for any positive continuous function ξ. This shows that $z^*(a) - ck^*(a) = 0$ for almost every $a = [0, \omega]$. Without loss of generality we can suppose that $z^* = ck^*$. By the definitions of z^*, η^*, and k^*,

$$x^*(a) = \eta^*(a) + ck^*(a) = \int_a^\omega [\eta(a) + ck(a)]\,da, \quad a \in [0, \omega].$$

Therefore, x^* is absolutely continuous with $x^*(\omega) = 0$, and

$$\dot{x}^*(a) = -\eta(a) - ck(a), \quad \text{a.e. } a \in [a, b]. \tag{3.7}$$

Substituting (3.7) for $x^*(a)$ in (3.3) we have, for $x \in C_1$,

$$\begin{aligned}\int_0^\omega \eta(a)x(a)\,da &= \int_0^\omega x^*(a)d_a x(a)\\&= -x^*(0)x(0) - \int_0^\omega \dot{x}^*(a)x(a)\,da\\&= -x^*(0)x(0) + \int_0^\omega \eta(a)x(a)\,da + c\int_0^\omega k(a)x(a)\,da\\&= -x^*(0)x(0) + \int_0^\omega \eta(a)x(a)\,da + cx(0).\end{aligned}$$

The above equality yields that $x^*(0) = c$. It follows that

$$\dot{x}^*(a) = -\eta(a) - x^*(0)k(a), \quad \text{a.e. } a \in [0, \omega]. \qquad \square$$

Proposition 3.3. *The formal adjoint operator* $\mathbf{A}^*$ *defined as above is a true adjoint operator of* $\mathbf{A}$.

Proof. Let $\tilde{\mathbf{A}}^*$ be the true adjoint operator of $\mathbf{A}$. For $\psi^* = (\psi_1^*, \ldots, \psi_n^*) \in D(\tilde{\mathbf{A}}^*)$, let $\tilde{\mathbf{A}}^*\psi^* = y^* = (y_1^*, \ldots, y_n^*) \in X^*$. Then $\langle \psi^*, \mathbf{A}\phi\rangle = \langle \tilde{\mathbf{A}}^*\psi^*, \phi\rangle = \langle y^*, \phi\rangle$ for all $\phi \in D(\mathbf{A})$. For $a \in [0, \omega]$ and $i = 1, \ldots, n$, we let

$$\eta_i(a) = -y_i^*(a) - \psi_i^*(a)\sigma_i(a) + \int_0^\omega \psi^*(s)B(s, a)\,ds.$$

Then $\eta_i \in L^\infty[0, \omega]$. In addition, for each fixed $i \in \{1, \ldots, n\}$ and any $\phi_i \in C_{1,i} = \{x \in C^1([0, \omega]);\ \int_0^\omega \beta_i(a)x(a) = x(0)\}$, we let $\phi = (\phi_1, \ldots, \phi_n)$ such that $\phi_j = 0$ for $j \neq i$. It is clear that $\phi \in D(A)$ and

$$\begin{aligned}(\mathbf{A}\phi)_i &= -\dot{\phi}_i - \sigma_i\phi_i + \int_0^\omega B_{ii}(\cdot, s)\phi_i(s)\,ds,\\(\mathbf{A}\phi)_j &= \int_0^\omega B_{ji}(\cdot, s)\phi_i(s)\,ds, \quad j \neq i.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^\omega y_i^*(a)\phi_i(a)\,da &= \langle y^*, \phi\rangle\\&= \langle \tilde{\mathbf{A}}^*\psi^*, \phi\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \psi^*, \mathbf{A}\phi \rangle \\
&= -\int_0^\omega \psi_i^*(a)\dot{\phi}_i(a)\,da \\
&\quad + \int_0^\omega \left[-\psi_i^*(a)\sigma_i(a) + \sum_{j=1}^n \int_0^\omega B_{ji}(s,a)\sigma_j^*(s)\,ds \right] \phi_i(a)\,da.
\end{aligned}
$$

From the equality above,

$$
\begin{aligned}
\int_0^\omega \psi_i^*(a)\phi_i(a) &= \int_0^\omega \left[-y_i^*(a) - \psi_i^*(a)\sigma_i(a) + \sum_{j=1}^n \int_0^\omega B_{ji}(s,a)\sigma_j^*(s)\,ds \right] \phi_i(a)\,da \\
&= \int_0^\omega \eta_i(a)\phi_i(a)\,da. \qquad (3.8)
\end{aligned}
$$

Since (3.8) holds for all $\phi_i \in C_{1,i}$, by Lemma 3.2, ψ_i^* is absolutely continuous with $\psi_i^*(\omega) = 0$, and

$$\dot{\psi}^* = -\eta_i - \psi_i^*(0)\beta_i \quad \text{a.e. on } [0,\omega].$$

By the definition of η_i,

$$y_i^* = \dot{\psi}_i^* - \sigma_i\psi_i^* + \psi_i^*(0)\beta_i + \sum_{j=1}^n \int_0^\omega B_{ji}(\cdot,s)\psi_j^*(a)\,da, \quad i = 1,\ldots,n. \qquad (3.9)$$

It follows from (3.9) and the definition of $\mathbf{A}^*$ that $y^* \in D(\mathbf{A}^*)$ and

$$A^*\psi^* = y^* = \tilde{\mathbf{A}}^*\psi^*.$$

It can also be easily verified that, if $\psi^* \in D(\mathbf{A}^*)$, then $\langle \mathbf{A}^*\psi^*, \phi \rangle = \langle \psi^*, \mathbf{A}\phi \rangle$ for all $\phi \in \mathbf{A}$. Therefore, $D(\mathbf{A}^*) = D(\tilde{\mathbf{A}}^*)$ and $\mathbf{A}^* = \tilde{\mathbf{A}}^*$. □

Proposition 3.4. *If $s(\mathbf{A}) > -\infty$, then $s(\mathbf{A})$ is an eigenvalue of both $\mathbf{A}$ and $\mathbf{A}^*$. In addition, Both $\mathbf{A}$ and $\mathbf{A}^*$ have a positive eigenfunction corresponding to $s(\mathbf{A})$.*

Proof. Let $S(t) : X \to X$ be the semigroup generated by the operator $\mathbf{A}_S : \mathcal{D}(\mathbf{A}) \to X$ given by

$$(\mathbf{A}_S\psi)_i(a) = -\dot{\psi}(a) - \sigma_i(a)\psi_i(a), \quad a \in [0,\omega], \quad i = 1,\ldots,n.$$

Note that the functions $\sigma(a)$ and $\beta(a)$ (see (3.1)) satisfy the Assumptions 5.1 and 5.2 in [14], respectively. It follows from Theorem 5.5 in [14] that $S(t)$ is eventually compact. Since $B_{ij} \in L^\infty([0,\omega]\times[0,\omega])$, we can choose a sequence $\{B_{ij}^m\}_{m=1}^\infty \subset C([0,\omega]\times[0,\omega])$ such that for all m and almost every $(a,s) \in ([0,\omega]\times[0,\omega])$,

$$0 \leqslant B_{ij}(a,s) \leqslant B_{ij}^{m+1}(a,s) \leqslant B_{ij}^m(a,s), \quad i,j=1,\dots,n$$

and

$$\lim_{m\to\infty}\int_0^\omega\int_0^\omega [B_{ij}^m(a,s)-B_{ij}(a,s)]\,ds\,da = 0, \quad i,j=1,\dots,n.$$

Let $\mathbf{B}^m : X \to X$ be defined as

$$(\mathbf{B}^m\psi)(a) = \int_0^\omega B^m(a,s)\psi(s)\,ds, \quad a\in[0,\omega],$$

where $B^m = [B_{ij}^m]$. It is clear that $\mathbf{B}^m$ is compact and $\mathbf{B}^{m+1} \leqslant \mathbf{B}^m$ for all m. Hence $\mathbf{A}_m = \mathbf{A}_S + \mathbf{B}^m$ is a compact perturbation of $\mathbf{A}_S$. It follows that $\mathbf{A}_m$ generates an eventually compact and positive semigroup for all m. Thus, $s(\mathbf{A}) \leqslant s(\mathbf{A}_{m+1}) \leqslant s(\mathbf{A}_m)$, and $s(\mathbf{A}_m)$ is an eigenvalue of $\mathbf{A}_m$ and $\mathbf{A}_m^*$, which is associated with a positive eigenfunction ϕ^m of $\mathbf{A}$ and a positive eigenfunction ϕ^{m*} of $\mathbf{A}_m^*$ for all m. Notice that $\phi^m \in C([0,\omega])$. Without loss of generality, we can suppose that $\|\phi^m\|_{C([0,\omega])} = 1$. Let $s^m = s(\mathbf{A}_m)$. By the equation $\mathbf{A}_m\phi^m = s^m\phi^m$ and the definition of $\mathbf{A}_m$,

$$\begin{aligned}\dot{\phi}_i^m(a) &= -[s^m+\sigma_i(a)]\phi_i^m(a) + \sum_{j=1}^n \int_0^\omega B_{ij}^m(a,s)\phi_j^m(s)\,ds,\\ i &= 1,\dots,n, \quad a\in[0,\omega].\end{aligned} \tag{3.10}$$

Thus, ϕ^m satisfies the equation

$$\begin{aligned}\phi_i^m(0) &= \int_0^\omega \beta_i(a)\phi_i^m(a)\,da,\\ \phi_i^m(a) &= \phi_i^m(0)e^{\int_0^a (s_m+\sigma_i(\theta))\,d\theta} + \sum_{j=1}^n \int_0^a e^{\int_\tau^a (s_m+\sigma_i(\theta))\,d\theta}\\ &\quad\times\int_0^\omega B_{ij}^m(\tau,s)\phi_j^m(s)\,ds\,d\tau, \quad a\in[0,\omega], \quad i=1,\dots,n.\end{aligned} \tag{3.11}$$

Since $\{s^m\}$ is monotonically decreasing and bounded below by $s(\mathbf{A})$, from Eq. (3.10), $\{\phi^m\}$ is pre-compact in $C([0,\omega])$. So, without loss of generality, we can suppose that

$\phi^m \to \phi \in C([0, \omega])$, and $s^m \to s_0 \geqslant s(\mathbf{A})$, as $m \to \infty$. By taking the limit as $m \to \infty$ in Eq. (3.11),

$$\begin{aligned} \phi_i(0) &= \int_0^\omega \beta_i(a)\phi_i(a)\,da, \\ \phi_i(a) &= \phi_i(0)e^{\int_0^a (s_0+\sigma_i(\theta))\,d\theta} + \sum_{j=1}^n \int_0^a e^{\int_\tau^a (s_0+\sigma_i(\theta))\,d\theta} \\ &\quad \times \int_0^\omega B_{ij}(\tau, s)\phi_j(s)\,ds\,d\tau, \quad a \in [0, \omega], \quad i = 1, \ldots, n. \end{aligned} \tag{3.12}$$

(3.12) yields that s_0 is an eigenvalue of $\mathbf{A}$ associated with a nonnegative eigenfunction ϕ. Thus, $s_0 \leqslant s(\mathbf{A})$. This, together with the inequality $s_0 \geqslant s(\mathbf{A})$, yields that $s_0 = s(\mathbf{A})$. By applying the same argument to the dual operator $\mathbf{A}_m^*$ one easily sees that $\mathbf{A}^*$ has an positive eigenvector ϕ^* associated with the eigenvalue $s(\mathbf{A})$. $\square$

Proposition 3.5. *If $\mathbf{A}$ is q-irreducible and $s(\mathbf{A}) > -\infty$, then $s(\mathbf{A})$ is a simple eigenvalue of $\mathbf{A}$.*

Proof. Let $s_0 = s(\mathbf{A})$ and let $\phi \geqslant 0$ be the eigenfunction of $\mathbf{A}$ corresponding to s_0. Then $\phi \gg 0$ for $\phi \notin \partial C_+$. Let ψ be any eigenfunction of $\mathbf{A}$ associated with s_0. Notice that both ϕ and ψ are continuous. Without loss of generality, we can suppose that $\phi \geqslant \psi$ and $\psi \nleqslant 0$ (otherwise we can obtain the desirable property by multiplying ϕ and ψ by suitable constants). Let $\alpha^* = \sup\{\alpha;\ \phi - \alpha\psi \geqslant 0\}$. The continuity of ϕ and ψ then implies that $\alpha^* \in \mathbb{R}$ and $\phi - \alpha^*\psi \in \partial C_+$. Moreover,

$$\mathbf{A}(\phi - \alpha^*\psi) = s_0(\phi - \alpha^*\psi).$$

If follows from the q-irreducibility of $\mathbf{A}$ that $\phi - \alpha^*\psi = 0$. This implies that $\phi = \alpha^*\psi$. Therefore, $\operatorname{Dim} \mathcal{N}(\mathbf{A} - s_0 I) = 1$. Next we shall show that

$$\mathcal{N}[(\mathbf{A} - s_0 I)^2] = \mathcal{N}(\mathbf{A} - s_0 I).$$

In fact, if there is a $\psi \in \mathcal{D}(\mathbf{A})\backslash\{0\}$ such that

$$(\mathbf{A} - s_0 I)^2\psi = 0,$$

then the fact that $\operatorname{Dim} \mathcal{N}(\mathbf{A} - s_0 I) = 1$ implies that $(\mathbf{A} - s_0 I)\psi = c\phi$ for some constant c. By Proposition 3.4, $\mathbf{A}^*$ has a positive eigenvector ϕ^* corresponding to s_0. We then have

$$0 = \langle(\mathbf{A}^* - s_0 I)\phi^*, \psi\rangle = \langle\phi^*, (\mathbf{A} - s_0 I)\psi\rangle = c\langle\phi^*, \phi\rangle.$$

It follows that $c = 0$ as $\langle \phi^*, \phi \rangle > 0$. Hence $\psi \in \mathcal{N}(\mathbf{A} - s_0 I)$, and hence $\mathcal{N}[(\mathbf{A} - s_0 I)^2] = \mathcal{N}(\mathbf{A} - s_0 I)$. $\square$

A direct consequence of Proposition 3.5 is that $\mathbf{A}^*$ has exactly one eigenfunction ϕ^* associated with $s(\mathbf{A}^*)$. Let

$$\omega_i = \min \left\{ a : \int_a^\omega \beta_i(\theta)\, d\theta = 0 \right\}, \quad i = 1, \ldots, n.$$

That is, $[0, \omega_i]$ is the support of β_i.

Proposition 3.6. *Suppose that* $\mathbf{A}$ *is q-irreducible and let* ϕ^* *be the nonnegative eigenfunction of* $\mathbf{A}^*$ *associated with* $s_0 = s(\mathbf{A})$. *Then the following hold*:

(1) *There are constants* $a_i^* \in [\omega_i, \omega]$, $i = 1, \ldots, n$, *such that*

$$\phi_i^*(a) > 0, \quad a \in [0, a_i^*),$$
$$\phi_i^*(a) = 0, \quad a \in [a_i^*, \omega].$$

(2) *Let* $X_{\mathbf{a}^*} = \{\varphi \in X : \varphi_i(a) = 0,\ a \in [0, a^*),\ i = 1, \ldots, n\}$. $X_{\mathbf{a}^*}$ *is invariant to* $T(t)$ *and* $\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}}) = 0$ *for* $t > 0$, *where* $\mathbf{r}(T)$ *denotes the spectral radius of the operator* T.

Proof. Using the expression of $\mathbf{A}^*$,

$$\dot{\phi}_i^*(a) = [\sigma_i(a) + s_0]\phi_i^*(a) - \beta_i(a)\phi_i^*(0) - \sum_{j=1}^n \int_0^\omega B_{ji}(s, a)\phi_j^*(s)\, ds$$
$$\text{a.e. } a \in [0, \omega] \tag{3.13}$$

for $i = 1, \ldots, n$. Applying the variation-of-constant formula to Eq. (3.13),

$$\begin{aligned} &\phi_i^*(a) e^{-\int_0^a (s_0 + \sigma_i(\theta))\, d\theta} \\ &\quad = \phi_i^*(0) - \int_0^a e^{-\int_0^\tau (s_0 + \sigma_i(\theta))\, d\theta} \\ &\qquad \times \left[\beta_i(\tau)\phi_i^*(0) + \sum_{j=1}^n \int_0^\omega B_{ji}(s, \tau)\phi_j^*(s)\, ds \right] d\tau. \end{aligned} \tag{3.14}$$

Noting that ϕ^* is nonnegative, (3.14) implies that $\phi_i^*(a) e^{-\int_0^a (s_0 + \sigma_i(\theta))\, d\theta}$ is decreasing. Using the fact that $\phi^*(\omega) = 0$, there exist $a_i^* \in [0, \omega]$, $i = 1, \ldots, n$, such that

$$\phi_i^*(a) > 0, \quad a \in [0, a_i^*),$$
$$\phi_i^*(a) = 0, \quad a \in [a_i^*, \omega].$$

Let $T^*(t)$ be the adjoint operator of $T(t)$. The restriction of $T^*(t)$ to the closure of $D(\mathbf{A}^*)$ is a C_0 semigroup with $\mathbf{A}^*$ being its infinitesimal generator [10, p. 39]. Hence $T^*(t)\phi^* = e^{s_0 t}\phi^*$ for all $t \geqslant 0$. It follows that for $t \geqslant 0$, if $0 \leqslant \psi \in X_{\mathbf{a}^*}$, then

$$\begin{aligned}\langle \phi^*, T(t)\psi\rangle &= \langle T^*(t)\phi^*, \psi\rangle \\ &= \langle e^{s_0 t}\phi^*, \psi\rangle \\ &= e^{s_0 t}\langle \phi^*, \psi\rangle \\ &= 0.\end{aligned}$$

Since $T(t)\psi \geqslant 0$ for $t \geqslant 0$, using the last equality,

$$[T(t)\psi](a) = 0, \quad a \in [0, a_i^*), \quad i = 1, \ldots, n.$$

Thus, $T(t)\psi \in X_{\mathbf{a}^*}$ for all $t \geqslant 0$, and $X_{\mathbf{a}^*}$ is invariant. Next, we claim that

$$\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}}) = 0, \quad t > 0. \tag{3.15}$$

Suppose on contrary that (3.15) is not true. Then,

$$\mathbf{r}(T(w)|_{X_{\mathbf{a}^*}}) > 0.$$

Let $\mathbf{A}|_{X_{\mathbf{a}^*}}$ be the restriction of $\mathbf{A}$ on $X_{\mathbf{a}^*}$. $\mathbf{A}|_{X_{\mathbf{a}^*}}$ is the infinitesimal generator of the semigroup $T(t)|_{X_{\mathbf{a}^*}}$. Therefore (see [6, Proposition 22, p. 251]),

$$s(\mathbf{A}|_{X_{\mathbf{a}^*}}) = \frac{1}{\omega}\ln(\mathbf{r}(T(w)|_{X_{\mathbf{a}^*}}) > -\infty.$$

Using the similar argument as in the proof of Proposition 3.4 one concludes that the operator $\mathbf{A}|_{X_{\mathbf{a}^*}}$ has a nonnegative eigenfunction associated with $s(\mathbf{A}|_{X_{\mathbf{a}^*}})$. That is, $\mathbf{A}$ has a nonnegative eigenfunction in the subspace $X_{\mathbf{a}^*}$. This contradicts the q-irreducibility of $\mathbf{A}$. Finally, let us show that $a_i^* \geqslant \omega_i$ for $i = 1, \ldots, n$. If this is not the case, without loss of generality, suppose that $a_i^* < \omega_1$. Since ϕ_1^* is nonnegative and $\phi_i^*(a) = 0$ for all $a \in [a_1^*, \omega]$, (3.14) yields that

$$\int_{a_1^*}^{\omega} \beta_1(\tau)\, d\tau \phi_1^*(0) = \int_{a_1^*}^{\omega} \left[\sum_{j=1}^{n} \int_0^{\omega} B_{j1}(s,\tau)\phi_j^*(s)\, ds \right] d\tau = 0. \tag{3.16}$$

Since $a_1^* < \omega_1$, using the definition of ω_1,

$$\int_{a_1^*}^{\omega} \beta_1(\tau)\, d\tau = \int_{a_1^*}^{\omega_1} \beta_1(\tau)\, d\tau > 0. \tag{3.17}$$

Eqs. (3.16) and (3.17) imply that $\phi_1^*(0) = 0$. It follows that $a_1^* = 0$. We define the operator $\bar{\mathbf{A}} : \mathcal{D}(\mathbf{A}) \to X$ by

$$[\bar{\mathbf{A}}\varphi]_i(a) = -\dot{\psi}_i(a) - \sigma_i(a)\varphi_i(a), \quad i = 1, \ldots, n.$$

Since $\int_0^{\omega} \beta_1(a)\, da > 0$, there is a $\lambda \in \mathbb{R}$ such that

$$\int_0^{\omega} \beta_1(a) e^{-\int_0^a (\sigma_1(\tau)+\lambda)\, d\tau}\, da = 1.$$

If we let $\psi = (\psi_1, \ldots, \psi_n) \in X_{\mathbf{a}^*}$ such that $\psi_1(a) = e^{-\int_0^a \sigma_1(\tau)\, d\tau}$ and $\psi_2 = \cdots = \psi_n = 0$, then it is obvious that $\psi \in \mathcal{D}(\mathbf{A}) \cap X_{\mathbf{a}^*}$ and $\bar{\mathbf{A}}\psi = \lambda\psi$. It follows that $s(\bar{\mathbf{A}}|_{X_{\mathbf{a}^*}}) > -\infty$. Noticing that $\mathbf{A} \geqslant \bar{\mathbf{A}}$, $s(\mathbf{A}|_{X_{\mathbf{a}^*}}) > -\infty$. Consequently, $\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}}) > 0$ for $t > 0$. This leads to a contradiction to (3.15). $\square$

Proposition 3.7. *Let* $\mathbf{A}$ *be q-irreducible and let* ϕ^* *be the nonnegative eigenfunction of* $\mathbf{A}^*$ *corresponding to* $s_0 = s(\mathbf{A})$. *For any* $\varphi \in X_+$, *if* $\langle \phi^*, \varphi \rangle > 0$, *then there is a* $t_0 > 0$ *such that* $u(t_0, \cdot, \varphi) \gg 0$.

Proof. Let $u(t,a) = (u_1(t,a), \ldots, u_n(t,a)) = u(t,a,\varphi)$, then $u_i(t,a)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) u_i(t,a) = -\sigma_i(a) u_i(t,a) + z_i(a,t), \tag{3.18}$$

where

$$z_i(a,t) = \sum_{j=1}^{n} \int_0^{\omega} B_{ij}(a,s) u_j(t,s)\, ds \geqslant 0, \quad a \in [0,\omega], \quad t \geqslant 0.$$

Claim. *For each fixed* $i = 1, \ldots, n$, *if there is a* $t_i \geqslant 0$ *such that*

$$\int_0^\omega \beta_i(a) u_i(t_i, a)\, da > 0,$$

then there is a $t_i^* > 0$ *such that* $u_i(t, \cdot) \gg 0$ *for all* $t \geqslant t_i^*$.

Proof of Claim. Let $v(t, a)$ be the solution of the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) v_i(t, a) = -\sigma_i(a) v_i(t, a)$$

satisfying the initial and boundary conditions

$$v(t, 0) = \int_0^\omega \beta_i(a) v(t, a)\, da, \quad v(0, a) = u_i(t_i, a), \quad a \in [0, \omega].$$

It is clear that $u_i(t + t_i, \cdot) \geqslant v(t, \cdot)$ for all $t \geqslant 0$. Since $\int_0^\omega \beta_i(a) v_i(0, a)\, da > 0$, the solution $v(t, \cdot)$ has asynchronous exponential growth [8]. That is, there is a $\lambda_0 \in \mathbb{R}$ and $c > 0$ such that

$$\lim_{t \to \infty} e^{-\lambda_0 t} v(t, \cdot) = c \hat{v}_i(\cdot)$$

in L^1 topology, where $\hat{v}_i(a) = e^{-\lambda_0 a - \int_0^a \sigma_i(\tau)\, d\tau}$, $a \in [0, \omega]$. Consequently,

$$\lim_{t \to \infty} e^{-\lambda_0 t} \int_0^\omega \beta_i(a) v_i(t, a)\, da = c \int_0^\omega \beta_i(a) \hat{v}_i(a)\, da > 0.$$

Thus, there exists a $T_i > \omega$ such that

$$u_i(t + t_i, 0) = \int_0^\omega \beta_i(a) u_i(t + t_i, a)\, da \geqslant \int_0^\omega \beta_i(a) v(t, a)\, da > 0$$

for all $t \geqslant T_i - \omega$. If we let $t_i^* = T_i + t_i$, then by solving (3.18) along its characteristic line we obtain that, for $t \geqslant t_i^*$ and $0 \leqslant a \leqslant \omega$,

$$u_i(t, a) = u_i(t - a, 0) e^{-\int_0^a \sigma_i(\tau)\, d\tau} + \int_0^a e^{-\int_s^a \sigma_i(\tau)\, d\tau} z(t - a + s, s)\, ds > 0.$$

This completes the proof of the *Claim*. $\square$

From the *Claim* above, to finish the proof of Proposition 3.7, it suffices to show that for each i, there is a $t_i \geqslant 0$ such that $\int_0^\omega \beta_i(a)u_i(t_i, a)\,da > 0$. Suppose that this is not the case. Then there is some i such that

$$\int_0^\omega \beta_i(a)[T(t)\varphi]_i(a)\,da = \int_0^\omega \beta_i(a)u_i(t_i, a)\,da = 0, \quad t \geqslant 0. \tag{3.19}$$

Let $\alpha = s_0 + 1$. For any positive integer n,

$$(\alpha I - \mathbf{A})^{-n}\varphi = \frac{1}{n-1}\int_0^\infty t^{n-1}e^{-\alpha t}T(t)\varphi\,dt.$$

Eq. (3.19) and the last equation yield that, for any positive integer n,

$$\begin{aligned}
&\int_0^\omega \beta_i(a)[(\alpha I - \mathbf{A})^{-n}\varphi]_i(a)\,da \\
&\quad = \frac{1}{n-1}\int_0^\omega \beta_i(a)\left[\int_0^\infty t^{n-1}e^{-\alpha t}T(t)\varphi\,dt\right]_i(a)\,da \\
&\quad = \frac{1}{n-1}\int_0^\infty t^{n-1}e^{-\alpha t}\left(\int_0^\omega \beta_i(a)[T(t)\varphi]_i(a)\,da\right)dt \\
&\quad = 0.
\end{aligned} \tag{3.20}$$

On the other hand, since s_0 is a simple eigenvalue of $\mathbf{A}$, by spectral mapping theorem,

$$\mathrm{ess} - \mathbf{r}((\alpha I - \mathbf{A})^{-1}) < \mathbf{r}((\alpha I - \mathbf{A})^{-1}) = \frac{1}{\alpha - s_0} = 1,$$

where $\mathrm{ess} - \mathbf{r}(T)$ denotes the essential spectral radius of an operator T. Let $X_1 = \{\psi \in X : \langle \phi^*, \psi \rangle = 0\}$. It is clear that X_1 is a closed subspace of X and is invariant to the operator $(\alpha I - \mathbf{A})^{-1}$. It follows that

$$\mathrm{ess} - \mathbf{r}((\alpha I - \mathbf{A})^{-1}|_{X_1}) \leqslant \mathrm{ess} - \mathbf{r}((\alpha I - \mathbf{A})^{-1}) < 1.$$

Using the inequality above we can show that

$$\mathbf{r}((\alpha I - \mathbf{A})^{-1}|_{X_1}) < 1. \tag{3.21}$$

If not, then $(\alpha I - \mathbf{A})^{-1}$ has an eigenvalue λ with $|\lambda| = 1$ associated with an eigenfunction $\xi \in CX_1$, which is the complex extension of X_1. Thus, $(\alpha I - \mathbf{A})^{-1}\xi = \lambda\xi$, or equivalently, $\mathbf{A}\xi = (\alpha - \frac{1}{\lambda})\xi = (s_0 + 1 - \bar{\lambda})\xi$. Notice that $\mathrm{Re}(s_0 + 1 - \bar{\lambda}) \leqslant s_0$ and $|\bar{\lambda}| = 1$. Hence $\bar{\lambda} = 1$ and ξ is an eigenfunction associated with s_0. This and $\xi \notin CX_1$ lead to a contradiction. Therefore, (3.21) holds.

Let ϕ be the positive eigenfunction of $\mathbf{A}$ associated with the eigenvalue $s(\mathbf{A})$. For any $\psi \in X$ we decompose ψ as

$$\psi = q\phi + \zeta,$$

where $q = \frac{\langle \phi^*, \psi \rangle}{\langle \phi^*, \phi \rangle} > 0$ and $\zeta \in X_1$. The facts that $(\alpha I - \mathbf{A})^{-1}\phi = \phi$ and $\mathbf{r}((\alpha I - \mathbf{A})^{-1}|_{X_1}) < 1$ yield that

$$\lim_{n\to\infty} (\alpha I - \mathbf{A})^{-n}\psi = \lim_{n\to\infty} (q(\alpha I - \mathbf{A})^{-n} + ((\alpha I - \mathbf{A})^{-n}|_{X_1})\zeta) = q\phi.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} \int_0^{\omega} \beta_i(a)[(\alpha I - \mathbf{A})^{-n}\psi]_i(a)\, da &= \int_0^{\omega} \beta_i(a) \lim_{n\to\infty} [(\alpha I - \mathbf{A})^{-n}\psi]_i(a)\, da \\ &= q\int_0^{\omega} \beta_i(a)\phi_i(a)\, da \\ &> 0.\end{aligned}$$

This contradicts (3.20) and the proof is completed. $\square$

Proposition 3.8. *Let $\hat{\mathbf{A}}$ be an infinitesimal generator obtained by replacing σ_i by $\hat{\sigma}_i$ $i = 1, \ldots, n$. Then the following hold*:

(1) *$\mathbf{A}$ and $\hat{\mathbf{A}}$ have the same q-reducibility.*
(2) *Suppose that $\mathbf{A}$ is q-irreducible. If ϕ^* and $\hat{\phi}^*$ are nonnegative eigenfunctions corresponding to eigenvalues $s(\mathbf{A})$ and $s(\hat{\mathbf{A}})$, respectively, then $\phi_i^*(a) = 0$ if and only if $\hat{\phi}_i^*(a) = 0$.*

Proof. To prove the statement (1) it is enough to show that if $\mathbf{A}$ is q-irreducible then $\hat{\mathbf{A}}$ is q-irreducible. We choose $v \in \mathbb{R}$ sufficiently large such that

$$v > \max_{1\leqslant i\leqslant n} \left\{ \|\sigma_i(\cdot) - \hat{\sigma}_i(\cdot)\|_{L^\infty[0,\omega]} \right\}.$$

Let $U(t,a) = e^{vt}T(t)\varphi$, $\varphi \in X_{\mathbf{a}^*}$, where $X_{\mathbf{a}^*}$ is defined in Proposition 3.6. $U(t,a) = (U_1(t,a), \ldots, U_n(t,a))$ satisfies the equations

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) U_i(t,a) \\ &\quad = -\hat{\sigma}_i(a)U_i(t,a) + \sum_{j=1}^{n} \int_{a_j}^{\omega} B_{ij}(a,s)U_j(t,s)\, ds \\ &\qquad + [v + \hat{\sigma}_i(a) - \sigma_i(a)]U_i(t,a), \quad t > 0, \quad a \in [a_i, \omega], \quad i = 1, \ldots, n,\end{aligned}$$

and

$$U_i(t,a)=0, \quad a\in[0,a_i), \quad U(0,\cdot)=\varphi, \quad i=1,\ldots,n.$$

Let $W(t,a)=(W_1(t,a),\ldots,W_n(t,a))$ with

$$W_i(t,a)=[v+\hat{\sigma}_i(a)-\sigma_i(a)]U_i(t,a).$$

It is clear that $W(t,\cdot)$ is nonnegative. Using the variation-of-constant formula,

$$e^{vt}T(t)\varphi=U(t,\cdot,\varphi)=\hat{T}(t)\varphi+\int_0^t \hat{T}(t-s)W(s,\cdot)\,ds\geqslant \hat{T}(t)\varphi$$

with $\hat{T}(t)$ being the semigroup generated by $\hat{\mathbf{A}}$. Similarly, $\hat{T}(t)\varphi\geqslant e^{-vt}T(t)\varphi$, for $\varphi\in X_{\mathbf{a}^*}$. Thus,

$$e^{vt}T(t)\varphi\geqslant\hat{T}(t)\varphi\geqslant e^{-vt}T(t)\varphi, \quad \varphi\in X_{\mathbf{a}^*}. \tag{3.22}$$

An immediate consequence of the inequality (3.22) is that $X_{\mathbf{a}^*}$ is invariant to $\hat{T}(t)$ and

$$\mathbf{r}(\hat{T}(t)|_{X_{\mathbf{a}^*}})\leqslant e^{vt}\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}})=0. \tag{3.23}$$

Next, let $\hat{\phi}$ be any nonnegative eigenfunction of $\hat{\mathbf{A}}$ and $\hat{\lambda}$ be the associated eigenvalue. Then one must have $\hat{\phi}\notin X_{\mathbf{a}^*}$, for otherwise $\mathbf{r}(\hat{T}(t)|_{X_{\mathbf{a}^*}})\geqslant e^{\hat{\lambda}t}$, a contradiction to (3.23). Thus, we have $\langle\phi^*,\hat{\phi}\rangle>0$. Rewrite $\hat{\phi}$ as

$$\hat{\phi}=\hat{q}\phi+\hat{\zeta}$$

with $\hat{q}=\frac{\langle\phi^*,\hat{\phi}\rangle}{\langle\phi^*,\phi\rangle}>0$ and $\hat{\zeta}\in X_{\mathbf{a}^*}$. It follows from (2) in Proposition 3.6 that

$$\lim_{t\to\infty} e^{-s_0t}T(t)\hat{\phi}=\hat{q}\phi.$$

The equality above and (3.22) yield that

$$e^{(-s_0+v)t}e^{-\hat{\lambda}t}\hat{\phi}=e^{(-s_0+v)t}\hat{T}(t)\hat{\phi}\geqslant e^{-s_0t}T(t)\hat{\phi}\to\hat{q}\phi\gg 0 \quad \text{as} \quad t\to\infty.$$

Consequently, $\hat{\phi}\gg 0$. Therefore, $\hat{\mathbf{A}}$ is q-irreducible. This complete the proof of (1).

Next, let $\hat{\phi}^*$ be the nonnegative eigenfunction of $\hat{\mathbf{A}}^*$ corresponding to the eigenvalue $\hat{s}_0 = s(\hat{\mathbf{A}})$ with

$$\hat{\phi}_i^*(a) > 0, \quad a \in [0, \hat{a}_i^*); \quad \hat{\phi}_i^*(a) = 0, \quad a \in [\hat{a}_i^*, \omega].$$

We claim that $a_i^* = \hat{a}_i^*$, $i = 1, \ldots, n$. If this is not true then either $X_{\mathbf{a}^*} \backslash X_{\hat{\mathbf{a}}^*} \neq \emptyset$ or $X_{\hat{\mathbf{a}}^*} \backslash X_{\mathbf{a}^*} \neq \emptyset$. Without loss of generality, suppose that $X_{\mathbf{a}} \backslash X_{\hat{\mathbf{a}}^*} \neq \emptyset$ and let $\xi \in X_{\mathbf{a}^*} \backslash X_{\hat{\mathbf{a}}^*}$. $\xi \in X_{\mathbf{a}^*}$ and $\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}}) = 0$ imply that

$$\lim_{\tau \to \infty} e^{(-\hat{s}_0 + v)t} T(t)\xi = 0.$$

On the other hand, $\langle \hat{\phi}^*, \xi \rangle > 0$ for $\xi \notin X_{\hat{\mathbf{a}}^*}$. It follows from (2) of Proposition 3.6 and (3.22) that

$$0 \ll \frac{\langle \hat{\phi}^*, \xi \rangle}{\langle \hat{\phi}^*, \hat{\phi} \rangle} \hat{\phi} = \lim_{t \to \infty} e^{-\hat{s}_0 t} \hat{T}(t)\xi \leqslant \lim_{t \to \infty} e^{(-\hat{s}_0 + v)t} T(t)\xi = 0.$$

This leads to a contradiction. □

We end this section with the following:

Proposition 3.9. *Suppose that* $\mathbf{A}$ *is q-irreducible. Let* $s_0 = s(\mathbf{A})$ *and let* $\mathbf{B} : X \to X$ *be defined by*

$$(\mathbf{B}\varphi)(a) = \int_0^a e^{-\int_s^a (\sigma(\tau) + s_0)\, d\tau} \left[\int_0^\omega B(s, \theta)\varphi(\theta)\, d\theta \right] ds, \quad a \in [0, \omega].$$

Then $(I - \mathbf{B})^{-1}$ *exists and is positive.*

Proof. First, we show that $\mathbf{r}(\mathbf{B}) < 1$. We observe that $\mathbf{B} : X \to X$ is compact and positive. If $r_0 = \mathbf{r}(\mathbf{B}) > 0$, then the dual operator $\mathbf{B}^*$ of $\mathbf{B}$ has an eigenvalue r_0 for which there exists some $\zeta^* \in X^* \backslash \{0\}$, $\zeta^* > 0$ such that

$$\mathbf{B}^* \zeta^* = r_0 \zeta^*.$$

Let $\phi \gg 0$ be the eigenfunction of $\mathbf{A}$ corresponding to s_0. Then $\phi(0) \gg 0$, and

$$\phi(a) = [\mathbf{B}\phi](a) + \exp\left(-\int_0^a [\sigma(\tau) + s_0]\, d\tau \right) \phi(0), \quad a \in (0, \omega],$$

or

$$[\mathbf{B}\phi](a) - r_0\phi(a) = (1 - r_0)\phi(a) - \exp\left(-\int_0^a [\sigma(\tau) + s_0]\, d\tau\right)\phi(0).$$

It follows that

$$0 = (1 - r_0)\langle\zeta^*, \phi\rangle - \left\langle\zeta^*, \exp\left(-\int_0^{\cdot} [\sigma(\tau) + s_0]\, d\tau\right)\phi(0)\right\rangle.$$

The last equality yields that

$$1 - r_0 = \frac{\langle\zeta^*, \exp(-\int_0^{\cdot} [\sigma(\tau) + s_0]\, d\tau)\phi(0)\rangle}{\langle\zeta^*, \phi\rangle} > 0$$

or $\mathbf{r}(\mathbf{B}) = r_0 < 1$. This implies that $I - \mathbf{B}$ is invertible. If we let $T_B(t)$ be the semigroup generated by $\mathbf{B}$, then $T_B(t)$ is positive, and for each $\varphi \in X_+$,

$$(I - \mathbf{B})^{-1}\varphi = \int_0^\infty e^{-t} T_B(t)\varphi\, dt \geqslant 0.$$

Hence $(I - \mathbf{B})^{-1}$ is positive. $\square$

4. Main theorems and proofs

In this section we give a characterization of the dynamics of (2.4).

Theorem 4.1. *If $s(\mathbf{A}) < 0$, then, for any solution $u(t, \cdot, \varphi)$ of* (2.4) *with initial function $\varphi \in X_+$,*

$$\lim_{t\to\infty} u(t, \cdot, \varphi) = 0.$$

Proof. Using the variation-of-constant formula,

$$u(t, \cdot, \varphi) = T(t)\varphi - \int_0^t T(t - s)Q(s)\, ds, \quad t \geqslant 0,$$

where $Q = (Q_1, \ldots, Q_n)$ with

$$Q_i(s) = \left[K_i(\cdot)u_i(s, \cdot, \varphi) + \int_0^{\omega} \sum_{j=1}^{n} K_{ij}(\cdot, \theta)u_j(s, \theta, \varphi)\, d\theta \right] u_i(s, \cdot, \varphi) \geqslant 0,$$

$$i = 1, \ldots, n.$$

From $s_0 = s(\mathbf{A}) < 0$ we know that, for any $\varphi \in X$,

$$\lim_{t\to\infty} T(t)\varphi = \lim_{t\to\infty} e^{s_0 t} \lim_{t\to\infty} e^{-s_0 t} T(t)\varphi = \lim_{t\to\infty} e^{s_0 t} \frac{\langle \phi^*, \varphi\rangle}{\langle \phi^*, \varphi\rangle} \varphi = 0,$$

where ϕ^* and ϕ are positive eigenfunctions of $\mathbf{A}^*$ and $\mathbf{A}$ corresponding to the eigenvalue s_0, respectively. The positivity of $T(t)$ therefore yield that

$$0 \leqslant u(t, \cdot, \varphi) \leqslant T(t)\varphi \to 0 \quad \text{as} \quad t \to \infty. \qquad \square$$

Theorem 4.2. *If* $\mathbf{A}$ *is q-irreducible and* $s(\mathbf{A}) > 0$, *then there exists a unique positive* (*endemic*) *equilibrium* u^+. *Furthermore, the positive equilibrium* u^+ *is globally stable in the following sense*: *let* $\phi^* > 0$ *be the eigenfunction of* $\mathbf{A}^*$, *the dual operator of* $\mathbf{A}$, *corresponding to* $s(\mathbf{A})$, *then, for any initial function* $\varphi \in X_+^{\Omega}$,

$$\lim_{t\to\infty} u(t, \cdot, \varphi) = u^+ \quad \textit{if} \ \langle \phi^*, \varphi\rangle > 0,$$
$$\lim_{t\to\infty} u(t, \cdot, \varphi) = 0 \quad \textit{if} \ \langle \phi^*, \varphi\rangle = 0.$$

To prove Theorem 4.2, we need some additional results. First, we rewrite (2.4) as an evolution equation

$$\frac{du(t, \cdot)}{dt} = F(u(t, \cdot)),$$

where $F : \mathcal{D}(\mathbf{A}) \to X$ is defined by

$$[F(\varphi)]_i(a) = [\mathbf{A}\varphi]_i(a) - \left[K_i(a)\varphi_i(a) + \sum_{j=1}^{n} \int_0^{\infty} K_{ij}(a, s)\varphi_j(s)\, ds \right] \varphi_i(s)\, ds.$$

Lemma 4.3. *If* $\mathbf{A}$ *is q-irreducible and* $s(\mathbf{A}) > 0$, *then* (2.4) *has at least one positive equilibrium* u^+. *Furthermore,* $u^+ \gg 0$.

Proof. Let $u(t,\psi) = u(t,\cdot,\psi)$ be the solution of (2.4). It is clear that if the initial function ψ is in $\mathcal{D}(\mathbf{A})$, then $u(t,\psi)$ is continuously differentiable for $t \geqslant 0$ and

$$\frac{du(t,\psi)}{dt} = F(u(t,\psi)).$$

Let $\phi^\varepsilon = \varepsilon\phi$, where $\phi \gg 0$ is the eigenfunction of $\mathbf{A}$ corresponding to $s_0 = s(\mathbf{A})$, and $\varepsilon > 0$ is a sufficiently small constant such that $\varepsilon\phi_i \leqslant p_i$, $i = 1, \ldots, n$, and

$$\varepsilon < \min \left\{ \frac{s_0}{\| K_i(\cdot)\phi_i(\cdot) + \sum_{j=1}^n \int_0^\omega K_{ij}(\cdot, s)\phi_j(s)\, ds \|_{L^\infty[0,\omega]}}; \quad i = 1, \ldots, n \right\}.$$

We can show that

$$\begin{aligned}
[F(\phi^\varepsilon)]_i(a) &= \mathbf{A}[(\phi^\varepsilon)]_i(a) - \left[K_i(a)\phi_i^\varepsilon(a) + \sum_{j=1}^n \int_0^\omega K_{ij}(a,s)\phi_j^\varepsilon(s)\, ds \right] \phi^\varepsilon(a) \\
&= \varepsilon \left(s_0 - \varepsilon \left[K_i(a)\phi_i(a) + \sum_{j=1}^n \int_0^\omega K_{ij}(a,s)\phi_j(s)\, ds \right] \right) \phi_i(a) \\
&> \delta\phi_i(a), \quad a \in [0,\omega], \quad i = 1, \ldots, n,
\end{aligned}$$

where δ is a positive number. Since $\phi^\varepsilon \in \mathcal{D}(\mathbf{A})$,

$$\left. \frac{du(t,\phi^\varepsilon)}{dt} \right|_{t=0} = F(\phi^\varepsilon) \gg 0.$$

Hence $u(t,\phi^\varepsilon)$ is increasing with respect to t for small t. The monotonicity of the flows introduced by (2.4) then implies that $u(t,\phi^\varepsilon)$ is increasing and $u(t,\phi^\varepsilon) \geqslant \phi^\varepsilon$ for all $t \geqslant 0$. Moreover, we have $u(t,p) \leqslant p$ for $t \geqslant 0$ and $\phi^\varepsilon < p$. It follows that $u(t,p)$ is decreasing and

$$\phi^\varepsilon \leqslant u(t,\phi^\varepsilon) \leqslant u(t,p) \leqslant p, \quad t \geqslant 0.$$

Therefore, Lebesgue's dominated convergence theorem implies that there is a $u^+ \in X_+^\Omega$ with $0 \ll \phi^\varepsilon \leqslant u^+ \leqslant p$ such that $u(t,p)$ converges to u^+. Therefore, $u^+ \gg 0$ is an equilibrium. $\square$

Lemma 4.4. *If* $\mathbf{A}$ *is q-irreducible and* $s(\mathbf{A}) > 0$, *then the nontrivial endemic equilibrium* u^+ *is unique.*

Proof. Let $\hat{u}^+$ be a nontrivial equilibrium with $0 \leqslant \hat{u}^+ \leqslant p$. Then $u(t,p) \geqslant \hat{u}^+$ for all $t \geqslant 0$. Consequently, $u^+ \geqslant \hat{u}^+$. We prove the uniqueness by showing that $u^+ = \hat{u}^+$. First, we see that $u^+ = (u_1^+, \ldots, u_n^+)$ and $\hat{u}^+ = (\hat{u}_1^+, \ldots, \hat{u}_n^+)$ satisfy the differential equations

$$\begin{aligned} \frac{du_i^+(a)}{da} &= -\vartheta_i(a)u_i^+(a) + \sum_{j=1}^n \int_0^\omega B_{ij}(a,s)u_j^+(s)\,ds, \\ \frac{d\hat{u}_i^+(a)}{da} &= -\hat{\vartheta}_i(a)\hat{u}_i^+(a) + \sum_{j=1}^n \int_0^\omega B_{ij}(a,s)\hat{u}_j^+(s)\,ds, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \vartheta_i(a) &= \mu_i(a) + \gamma_i(a) - p_i(a)K_i(a) \\ &\quad + K_i(a)u_i^+(a) + \sum_{j=1}^n \int_0^\omega K_{ij}(a,s)u_j^+(s)\,ds, \\ \hat{\vartheta}_i(a) &= \mu_i(a) + \gamma_i(a) - p_i(a)K_i(a) \\ &\quad + K_i(a)\hat{u}_i^+(a) + \sum_{j=1}^n \int_0^\omega K_{ij}(a,s)\hat{u}_j^+(s)\,ds, \quad i = 1, \ldots, n. \end{aligned} \tag{4.2}$$

By (4.2) and the inequality $u^+ \geqslant \hat{u}^+$, $\vartheta_i \geqslant \hat{\vartheta}_i$ for $i = 1, \ldots, n$.

Claim 1. *Let $a_1^*, \ldots, a_n^*$ be defined as in Proposition 3.4. Then $\vartheta_i(a) = \hat{\vartheta}_i(a)$ for $a \in [0, a_i^*)$ and for $i = 1, \ldots, n$.*

Proof of Claim 1. Let $\mathbf{A}_+, \hat{\mathbf{A}}_+ : \mathcal{D}(\mathbf{A}) \to X$ be defined, respectively, by

$$[\mathbf{A}_+\varphi]_i(a) = -\dot{\varphi}_i(a) - \vartheta_i(a)\varphi_i(a) + \sum_{j=1}^n \int_0^\omega B_{ij}(a,s)\varphi_j(s)\,ds,$$

$$[\hat{\mathbf{A}}_+\varphi]_i(a) = -\dot{\varphi}_i(a) - \hat{\vartheta}_i(a)\varphi_i(a) + \sum_{j=1}^n \int_0^\omega B_{ij}(a,s)\varphi_j(s)\,ds$$

for $i = 1, \ldots, n$. Thus, by Proposition 3.8, both $\mathbf{A}_+$ and $\hat{\mathbf{A}}_+$ are q-irreducible. Also by definitions of $\hat{\mathbf{A}}_+$,

$$\mathbf{A}_+u^+ = 0, \quad \hat{\mathbf{A}}_+\hat{u}^+ = 0.$$

Thus, the q-irreducibility of $\hat{\mathbf{A}}_+$ implies that $\hat{u}^+$ is strictly positive. One can easily show that

$$s(\mathbf{A}_+) = s(\hat{\mathbf{A}}_+) = 0. \tag{4.3}$$

Let ξ^* be the nonnegative eigenfunction of $\mathbf{A}_+^*$ corresponding to $s(\mathbf{A}_+) = 0$ and let ϕ^* be the nonnegative eigenfunction of $\mathbf{A}^*$ corresponding to $s(\mathbf{A})$. It follows from Proposition 3.8 that ξ^* and ϕ^* have the same support. That is,

$$\xi_i^*(a) > 0, \quad a \in [0, a_i^*); \quad \xi_i^*(a) = 0, \quad a \in [a_i^*, \omega], \quad i = 1, \ldots, n.$$

By the definitions of $\mathbf{A}_+$ and $\hat{\mathbf{A}}_+$,

$$[\mathbf{A}_+\hat{u}^+]_i(a) = [\hat{\mathbf{A}}_+\hat{u}^+]_i(a) - \hat{u}_i^+(a)[\vartheta_i(a) - \hat{\vartheta}_i(a)] = -\hat{u}_i^+(a)[\vartheta_i(a) - \hat{\vartheta}_i(a)],$$
$$a \in [0, \omega]$$

for $i = 1, \ldots, n$. It follows from the equality above that

$$0 = \langle A_+^*\xi^*, \hat{u}^+\rangle = \langle \xi^*, \mathbf{A}_+\hat{u}^+\rangle = -\sum_{i=1}^n \int_0^{a_i^*} \xi_i^*(a)\hat{u}_j^+(a)[\vartheta_i(a) - \hat{\vartheta}_i(a)]\, da.$$

Noticing that

$$\vartheta_i(a) - \hat{\vartheta}_i(a) \geqslant 0, \quad \xi_i^*(a)u_i^+(a) > 0, \quad a \in [0, a_i^*),$$

we have

$$\vartheta_i(a) - \hat{\vartheta}_i(a) = 0, \quad a \in [0, a_i^*), \quad i = 1, \ldots, n.$$

This completes the proof of *Claim* 1. $\square$

Claim 2. $u^+(0) = \hat{u}^+(0)$.

Proof of Claim 2. We prove this claim by contradiction. Suppose that $u^+(0) \neq \hat{u}^+(0)$. Then there is a k with $1 \leqslant k \leqslant n$ such that $u_k^+(0) \neq \hat{u}_k^+(0)$. By Claim 1 and (4.2) one can easily deduce that, for $a \in [0, a_k^*)$,

$$\sum_{j=1}^n \int_0^\omega K_{kj}(a, s)u_j^+(s)\, ds = \sum_{j=1}^n \int_0^\omega K_{kj}(a, s)\hat{u}_j^+(s)\, ds.$$

From (3.1),

$$\sum_{j=1}^{n}\int_0^{\omega} B_{kj}(a,s)u_j^+(s)\,ds=\sum_{j=1}^{n}\int_0^{\omega} B_{kj}(a,s)\hat{u}_j^+(s)\,ds. \tag{4.4}$$

Thus, from (4.1), (4.4) and Claim 1 it follows that

$$\frac{d}{da}[u_k^+(a)-\hat{u}_k^+(a)]=-\vartheta_k(a)[u_k^+(a)-\hat{u}_k^+(a)],\quad a\in[0,a_k^*).$$

Therefore, for $a\in[0,a_k^*)$,

$$u_k^+(a)-\hat{u}_k^+k(a)=[u_k^+(0)-\hat{u}_k^+(0)]e^{-\int_0^a \vartheta_k(\tau)\,d\tau}. \tag{4.5}$$

By Proposition 2.6, $\beta_k(a)=0$ for $a\in(\omega_k,\omega]$ and $a_k^*\geqslant\omega_k$. By using (4.5) and the boundary conditions on u_k^+ and $\hat{u}_k^+$,

$$\begin{aligned}u_k^+(0)-\hat{u}_k^+(0)&=\int_0^{\omega_k}\beta_k(a)[u_k^+(a)-\hat{u}_k^+(a)]\,da\\&=\int_0^{\omega_k}\beta_k(a)e^{-\int_0^a \vartheta_k(\tau)\,d\tau}\,da[u_k^+(0)-\hat{u}_k^+(0)].\end{aligned}$$

Therefore,

$$\int_0^{\omega_k}\beta_k(a)e^{-\int_0^a \vartheta_k(\tau)\,d\tau}\,da=1. \tag{4.6}$$

On the other hand, (4.1) yields that

$$u_k^+(a)=e^{-\int_0^a \vartheta_k(\tau)\,d\tau}u_k^+(0)+y_k(a),\quad a\in[0,a_k^*),$$

with

$$y_k(a)=\int_0^a e^{-\int_s^a \vartheta_k(\tau)\,d\tau}\left[\int_0^{\omega}\sum_{j=1}^{n}B_{kj}(s,\theta)u_j^+(\theta)\,d\theta\right]ds.$$

It follows from the boundary condition to u_k^+ that

$$u_k^+(0)=\int_0^{\omega_k}\beta_k(a)e^{-\int_0^a \vartheta_k(\tau)\,d\tau}\,da\,u_k^+(0)+\int_0^{\omega_k}\beta_k(a)y_k(a)\,da.$$

The last equality and (4.6) yield that

$$\int_0^{\omega_k} \beta_k(a) y_k(a)\, da = 0.$$

From the expression of $y_k(a)$ and the definition of ω_k, $y_k(a) = 0$ for $a \in [0, \omega_k]$. Consequently,

$$\int_0^{\omega} \sum_{j=1}^{n} K_{kj}(a,s) u_j^+(s)\, ds = 0, \quad a \in [0, \omega_k].$$

Therefore, by the strict positivity of u_i^+,

$$\int_0^{\omega} \sum_{j=1}^{n} K_{kj}(a,s)\, ds = 0, \quad a \in [0, \omega_k]. \tag{4.7}$$

Moreover, it follows from $u_k^+(a) > \hat{u}_k^+(a)$ and $\vartheta_k(a) = \hat{\vartheta}_k(a)$ that

$$K_i(a) = 0, \quad a \in [0, a_k^*). \tag{4.8}$$

Let $\phi = (\phi_1, \ldots, \phi_n)$ be the positive eigenfunction of $\mathbf{A}$ associated with $s_0 = s(\mathbf{A}) > 0$. From (4.7), (4.8), and the definitions of σ_k and ϑ_k we deduce that

$$\sigma_k(a) = \mu_k(a) + \gamma_k(a) = \vartheta_k(a), \quad a \in [0, \omega_k] \tag{4.9}$$

and

$$\dot{\phi}_k(a) = -(\sigma_k(a) + s_0)\phi_k(a), \quad a \in [0, \omega_k).$$

Thus,

$$\phi_k(a) = \phi_k(0) e^{-\int_0^a [\sigma_k(\tau) + s_0]\, d\tau}, \quad a \in [0, \omega_k).$$

It follows from the equality above, (4.6), (4.9), and the boundary condition of ϕ_k that

$$1 = \int_0^{\omega_k} \beta_k(a) e^{-\int_0^a [\sigma_k(\tau)+s_0]\, d\tau}\, da < \int_0^{\omega_k} \beta_k(a) e^{-\int_0^a \sigma_k(\tau)\, d\tau}\, da = 1.$$

This is a contradiction and, therefore, Claim 2 holds.

By applying the variation-of-constant formula to (4.1) and using Claim 2,

$$u^+(a) - \hat{u}^+(a) = (\mathbf{B}_+[u^+ - \hat{u}^+])(a) - \int_0^a \exp\left(-\int_\theta^a \vartheta(\tau)\,d\tau\right) W(\theta)\,d\theta, \tag{4.10}$$

where $\mathbf{B}_+ : X \to X$ is defined by

$$(\mathbf{B}_+\varphi)(a) = \int_0^a e^{-\int_s^a \vartheta(\tau)\,d\tau} \left[\int_0^\omega B(s,\theta)\varphi(\theta)\,d\theta\right] ds, \quad a \in [0,\omega].$$

$\vartheta = \mathrm{diag}(\vartheta_i)$, and $W(a) = (W_1(a), \ldots, W_n(a))$ with

$$W_i(a) = [\vartheta_i(a) - \hat{\vartheta}_i(a)]\hat{u}_i^+(a) \geqslant 0.$$

Since $s(\mathbf{A}_+) = 0$, by applying Proposition 3.8 to the operators $\mathbf{A}_+$ and $\mathbf{B}_+$ we see that $(I - \mathbf{B}_+)^{-1}$ exists and that it is a positive operator. Therefore, from $W \geqslant 0$ and (4.10),

$$0 \leqslant u^+ - \hat{u}^+ = -(I - \mathbf{B}_+)^{-1}\left[\int_0^\cdot \exp\left(-\int_\theta^\cdot \sigma(\tau)\,d\tau\right) W(\theta)\,d\theta\right] \leqslant 0.$$

It follows that $u^+ = \hat{u}^+$. $\square$

We are now in the position to give the following proof.

Proof of Theorem 4.2. First, if $\varphi \in X_+$ and $\langle \phi^*, \varphi\rangle = 0$, then $\varphi \in X_{\mathbf{a}^*}$. Therefore, $\mathbf{r}(T(t)|_{X_{\mathbf{a}^*}}) = 0$ (see (2) of Proposition 3.6) implies that $T(t)\varphi \to 0$ as $t \to \infty$. The variation-of-constant formula yields that

$$0 \leqslant u(t,\cdot,\varphi) \leqslant T(t)\varphi \to 0 \quad \text{as} \quad t \to \infty,$$

and hence, $\lim_{t\to\infty} u(t,\cdot,\varphi) = 0$.

Next, if $\varphi \in X_+^\Omega$ with $\langle \phi^*, \varphi\rangle > 0$, then by Proposition 3.7 there is a $t_0 > 0$ such that $u(t_0,\cdot,\varphi) \gg 0$. As we did in the proof of Lemma 4.3, we can choose an $\varepsilon > 0$ sufficiently small such that

$$F(\phi^\varepsilon) \gg 0 \quad \text{and} \quad \phi^\varepsilon \leqslant u^+(t_0,\cdot,\varphi).$$

Therefore, by the monotonicity,

$$u(t,\cdot,\phi^\varepsilon) \leqslant u(t+t_0,\cdot,\varphi) \leqslant u(t,\cdot,p). \tag{4.11}$$

From Lemma 4.4, $u(t,\cdot,\phi^{\varepsilon})$ and $u(t,\cdot,p)$ converge to the unique positive equilibrium u^+ as $t\to\infty$. Hence (4.11) yields that $\lim_{t\to\infty} u(t,\cdot,\varphi)=u^+$. $\square$

Remark. We have not discussed the case of $s(\mathbf{A})=0$ in this paper. However, we point out that when $s(\mathbf{A})=0$, (2.4) cannot have a positive equilibrium and hence the zero solution is globally stable. The proof requires the use of some further properties of irreducibility of the operator $\mathbf{A}$. We omit the proof in order to maintain the paper in a reasonable length.

5. An example

In this section, we consider an example in which the kernel functions are separable. In a more general sense, suppose that K_{ij} satisfies the following properties:

(a) There are nonnegative functions $H^1_{ij}(a)$, $H^2_{ij}(a)$ and an $\varepsilon>0$ such that

$$\varepsilon H^1_{ij}(a)H^2_{ij}(s)\leqslant K_{ij}(a,s)\leqslant H^1_{ij}(a)H^2_{ij}(s),\quad i,j=1,\ldots,n.$$

(b) $\operatorname{sign}\left(\int_0^\omega \beta_i(a)\int_0^a H^1_{ij}(t)\,dt\,da\right)=\operatorname{sign}\left(\int_0^\omega\int_0^\omega K_{ij}(a,s)\,ds\,da\right),\quad i,j=1,\ldots,n.$

(c) The matrix $\mathbf{K}=\left(\int_0^\omega\int_0^\omega K_{ij}(a,s)\,da\,ds\right)_{n\times n}$ is irreducible.

Theorem 5.1. *Under the assumptions* (a)–(c) *the following hold*:

(i) *If* $s(\mathbf{A})<0$, *then* $\lim_{t\to\infty}u(t,\cdot,\varphi)=0$ *for each* $\varphi\in X_+$.

(ii) *If* $s(\mathbf{A})>0$ *then System* (2.4) *has a unique* (*strictly*) *positive equilibrium* u^+ *such that*

$$\lim_{t\to\infty}u(t,\cdot,\varphi)=u^+,\quad \text{if}\ \sum_{i=1}^n\int_0^{a_i^*}\varphi_i(a)\,da>0,$$

$$\lim_{t\to\infty}u(t,\cdot,\varphi)=0,\quad \text{if}\ \sum_{i=1}^n\int_0^{a_i^*}\varphi_i(a)\,da=0,$$

where

$$a_i^*=\min\left\{a:\int_a^\omega\left[\beta_i(s)+\sum_{j=1}^n H^2_{ji}(s)\right]ds=0\right\},\quad i=1,\ldots,n.$$

Proof. By Theorem 4.2 we see that, to prove Theorem 5.1, we only need to show that the operator $\mathbf{A}$ is q-irreducible under assumptions (a)–(c), and that the nonnegative eigenfunction $\phi^* = (\phi_1^*, \ldots, \phi_n^*)$ of $\mathbf{A}^*$ corresponding to $s(\mathbf{A})$ has the support $[0, a_i^*]$ for each component ϕ_i^*. Let us first show that if $\phi = (\phi_1, \ldots, \phi_n)$ is any nonnegative eigenfunction of $\mathbf{A}$, then $\phi \gg 0$. Let λ be the corresponding eigenvalue.

$$\begin{aligned}\phi_i(a) &= e^{-\int_0^a [\sigma_i(\tau)+\lambda]\, d\tau} \phi_i(0) \\ &\quad + \int_0^a e^{-\int_t^a [\sigma_i(\tau)+\lambda]\, d\tau} \sum_{j=1}^n \int_0^\omega p_i(t) K_{ij}(t,s) \phi_j(s)\, ds\, dt. \end{aligned} \tag{5.1}$$

It follows from Eq. (5.1) that, if $\phi_i(0) \neq 0$, then $\phi_i \gg 0$. If $\phi_i(0) = 0$, applying the boundary condition to ϕ_i,

$$0 = \int_0^\omega \beta_i(a) \int_0^a \left[e^{-\int_t^a [\sigma_i(\tau)+\lambda]\, d\tau} \sum_{j=1}^n \int_0^\omega p_i(t) K_{ij}(t,s) \phi_j(s)\, ds\, dt \right] da.$$

By using the assumption (a) we deduce that

$$\left(\int_0^\omega \beta_i(a) \int_0^a H_{ij}^1(t)\, dt\, da \right) \left(\int_0^\omega H_{ij}^2(s) \phi_j(s)\, ds \right) = 0, \quad j = 1, \ldots, n.$$

From the assumption (b),

$$\int_0^\omega H_{ij}^2(s) \phi_j(s)\, ds = 0, \quad j = 1, \ldots, n.$$

It follows that

$$\int_0^\omega \sum_{j=1}^n \int_0^\omega p_i(a) K_{ij}(a,s) \phi_j(s)\, ds\, da = 0, \quad j = 1, \ldots, n.$$

Hence $\phi_i \equiv 0$. We have shown that, for each i, either $\phi_i \gg 0$ or $\phi_i \equiv 0$. If ϕ is not strictly positive, then, without loss of generality, by rearranging the order of the components, we can assume that

$$\phi_1 = \cdots = \phi_l = 0, \quad \phi_m \gg 0, \quad m = l+1, \ldots, n.$$

Thus, (5.1) yields that

$$\int_0^\omega \int_0^\omega K_{ij}(a,s)\, da\, ds = 0, \quad i = 1, \ldots, l, \quad j = l+1, \ldots, n.$$

This shows that the matrix $\mathbf{K}$ defined in (c) is reducible, a contradiction.

Next, let ϕ^* be the nonnegative eigenfunction of $\mathbf{A}^*$ corresponding to $s_0 = s(\mathbf{A})$ and let $\phi_i^*(a) > 0$, $a \in [0, \bar{a}_i)$ and $\phi_i^*(a) = 0$, $a \in [\bar{a}_i, \omega]$, $i = 1, \ldots, n$. We claim that $\bar{a}_i \geqslant a_i^*$ for $i = 1, \ldots, n$. To see this, we use the equality (3.14) in Section 3 to obtain

$$\int_{\bar{a}_i}^{\omega} e^{-\int_0^s [\sigma_i(\theta)+s_0]\, d\theta} \left[\beta_i(\tau)\phi_i^*(0) + \sum_{j=1}^{n} \int_0^{\omega} p_j(t) K_{ji}(t,s) \phi_j^*(t)\, dt \right] ds = 0,$$

for $i = 1, \ldots, n$. It follows that

$$\int_{\bar{a}_i}^{\omega} \beta_i(s)\, ds = \int_{\bar{a}_i}^{\omega} \left[\sum_{j=1}^{n} \int_0^{\bar{a}_j} K_{ji}(t,s)\phi_j^*(t)\, dt \right] ds = 0, \quad i = 1, \ldots, n.$$

From the assumption (a) we deduce that

$$\int_0^{\bar{a}_j} H_{ji}^1(t)\, dt \int_{\bar{a}_i}^{\omega} H_{ij}^2(s)\, ds = 0, \quad i, j = 1, \ldots, n. \tag{5.2}$$

The fact that $\int_{\bar{a}_j}^{\omega} \beta_j(s)\, ds = 0$ (by Proposition 3.6), $j = 1, \ldots, n$, yields that

$$\int_0^{\omega} \beta_j(a) \int_0^a H_{ji}^1(t)\, dt\, da = \int_0^{\bar{a}_j} \int_0^a H_{ji}^1(t)\, dt\, da.$$

Since $\int_0^a H_{ji}^1(t)\, dt$ is nonnegative and increasing, from the equality above,

$$\int_0^{\bar{a}_j} H_{ji}^1(t)\, dt > 0 \quad \text{whenever} \quad \int_0^{\omega} \beta_j(a) \int_0^a H_{ji}^1(t)\, dt > 0. \tag{5.3}$$

Moreover, from the assumption (b),

$$\int_0^{\omega} \beta_j(a) \int_0^a H_{ji}^1(t)\, dt > 0 \quad \text{whenever} \quad \int_{\bar{a}_i}^{\omega} H_{ji}^2(s)\, ds > 0. \tag{5.4}$$

By combining (5.2)–(5.4),

$$\int_{\bar{a}_i}^{\omega} H_{ij}^2(s)\, ds = 0, \quad i, j = 1, \ldots, n.$$

It follows from the definition of a_i^* that

$$\bar{a}_i \geqslant a_i^*, \quad i = 1, \ldots, n. \tag{5.5}$$

Furthermore, from the definitions of $a_1^*, \dots, a_n^*$ we have

$$\int_0^{a_i^*} \left[\int_{a_j^*}^{\omega} K_{ij}(a,s)\,ds \right] da = 0, \quad i,j = 1,\dots,n.$$

Using the equality above, it is not difficult to verify that the set

$$X_{\mathbf{a}^*} = \{\varphi \in X_+ : \varphi_i(a) = 0,\ a \in [0, a_i^*),\ i = 1,\dots,n\}$$

is invariant to the semigroup $T(t)$. Since $\mathbf{A}$ is q-irreducible, if we let $\varphi \in X_{\mathbf{a}^*}$ such that $\varphi_i(a) = 1$, $a \in [a_i^*, \omega]$, then,

$$\lim_{t\to\infty} e^{-s_0 t} T(t)\varphi = 0.$$

On the other hand, from Proposition 3.7,

$$\lim_{t\to\infty} e^{-s_0 t} T(t)\varphi = \frac{\langle \phi^*, \varphi\rangle}{\langle \phi^*, \phi\rangle}\,\phi.$$

Therefore, (5.5) yields that

$$0 = \langle \phi^*, \varphi\rangle = \sum_{i=1}^{n} \int_0^{\omega} \phi_i^*(a)\varphi_i(a)\,da = \sum_{i=1}^{n} \int_{a_i^*}^{\bar{a}_i} \phi_i^*(a)\,da.$$

Thus, we must have $a_i^* = \bar{a}_i$, $i = 1,\dots,n$. $\square$

Remark. If $n = 1$, then we obtain the result in [3] for a single group model.

References

[2] S. Busenberg, K. Cooke, Vertically Transmitted Diseases, Biomathematics, vol. 23, Springer, Berlin, 1993.

[3] S. Busenberg, M. Iannelli, H. Thieme, Global behavior of an age structured S–I–S epidemic model, SIAM J. Math. Anal. 22 (1991) 1065–1080.

[4] S. Busenberg, M. Iannelli, H. Thieme, Dynamics of an age structured epidemic model, in: S.T. Liao, Y.Q. Ye, T.R. Ding (Eds.), Dynamical Systems, Nankai Series in Pure, Applied Mathematics and Theoretical Physics, vol. 4, World Scientific, Singapore, 1993, pp. 1–19.

[6] K.-J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, New York, 2000.

[7] G. Folland, Real Analysis, Wiley, New York, 1984.

[8] H. Inaba, A semigroup approach to the strong ergodic theorem of the multi-state stable population process, Math. Popul. Stud. 1 (1) (1988) 49–77.
[9] A. Lajmanovich, J. York, A deterministic model for gonorrhea in a nonhomogeneous population, Math. Biosci. 28 (1997) 221–236.
[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44, Springer, Berlin, Heidelberg, New York, 1983.
[11] J. PrüB, Equilibrium Solutions of age-specific population dynamics of several species, J. Math. Biol. 11 (1981) 65–84.
[12] H.L. Smith, Systems of ordinary differential equations which generate an order preserving flow, SIAM Rev. 30 (1988) 87–113.
[14] H.R. Thieme, Positive perturbation of operator semigroups: growth bounds, essential compactness, and asynchronous exponential grough, Discrete Continuous Dynamical Systems 4 (1998) 735–764.
[15] G. Webb, Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, New York, Basel, 1985.
[16] E. Zeidler, Nonlinear Functional Analysis and its Applications I, Springer, New York, 1988.