Lesson 38. Poisson formula

The Poisson formula enables us to solve the boundary value problem $\nabla^2\Phi = 0$ in the unit disk, with prescribed values $\Phi(e^{it})$ on the boundary:

$$\Phi(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it})P(r, t-\theta)\,\mathrm{d}t$$

where $P(r,\theta) = \dfrac{1-r^2}{1-2r\cos\theta + r^2}$ is the **Poisson kernel**.

Using conformal mappings, this solves a boundary value problem in any domain D for which a conformal mapping of D onto the unit disk is known.

Example. For a harmonic function Φ in a disk $|z| < R$,

$$\Phi(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(Re^{it})\frac{R^2-r^2}{R^2-2Rr\cos(t-\theta)+r^2}\,\mathrm{d}t.$$

Example. For a harmonic function Φ in the upper half plane $y > 0$,

$$\Phi(x+iy) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\Phi(t+i0)}{(x-t)^2+y^2}\,\mathrm{d}t.$$

The Poisson formula in the unit disk can be derived in terms of Fourier series, as $r^n \cos n\theta = \operatorname{Re} z^n$ and $r^n \sin n\theta = \operatorname{Im} z^n$ are harmonic. Thus, if

$$\Phi(e^{it}) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

then $\Phi(re^{i\theta}) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$.

Here we prove it using complex integration.

To see the connection with Cauchy's formula, let $\Psi(z)$ be a harmonic conjugate of $\Phi(z)$, so that $F(z) = \Phi(z) + i\Psi(z)$ is analytic in the unit disk $|z| < 1$. We assume F to be continuous in the closed disk $|z| \leq 1$. Then, $F(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{F(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \int_0^{2\pi} \dfrac{\zeta F(\zeta)}{\zeta - z}\,\mathrm{d}t$.
Here C is the unit circle traversed counterclockwise, $\zeta = e^{it}$, and $\mathrm{d}\zeta = ie^{it}\,\mathrm{d}t = i\zeta\,\mathrm{d}t$.

Since $\frac{1}{\bar{z}}$ is outside C, replacing z by $\frac{1}{\bar{z}}$ in the integral we get, by Cauchy's Theorem,

$$0 = \frac{1}{2\pi i}\int_C \frac{F(\zeta)}{\zeta - \frac{1}{\bar{z}}}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta F(\zeta)}{\zeta - \frac{1}{\bar{z}}}\,\mathrm{d}t =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\zeta F(\zeta)}{\zeta - \frac{\zeta\bar{\zeta}}{\bar{z}}}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \frac{\bar{z}F(\zeta)}{\bar{z} - \bar{\zeta}}\,\mathrm{d}t.$$

Subtracting these two integrals and using, for $\zeta = e^{it}$, $z = re^{i\theta}$, and $\zeta - z = e^{it}(1 - re^{i(t-\theta)})$,

$$\frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\bar{\zeta} - \bar{z}} = \frac{\zeta\bar{\zeta} - z\bar{z}}{(\zeta - z)(\bar{\zeta} - \bar{z})} = \frac{1 - r^2}{1 - 2r\cos(t - \theta) + r^2},$$

we get $F(re^{i\theta}) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F(e^{it})P(r, t - \theta)\,\mathrm{d}t$.

Taking real parts we get

$$\Phi(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it})P(r, t - \theta)\,\mathrm{d}t.$$

There are some useful identities for the Poisson kernel: for $z = re^{i\theta}$ and $r < 1$,

$$\operatorname{Re} \frac{1+z}{1-z} = \operatorname{Re} \left(1 + 2 \sum_{n=1}^{\infty} z^n \right) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Also,

$$\operatorname{Re} \frac{1+z}{1-z} = \operatorname{Re} \frac{(1+re^{i\theta})(1-re^{i\theta})}{|1-re^{i\theta}|^2} =$$

$$\operatorname{Re} \frac{1-r^2+2ir\sin\theta}{1-2r\cos\theta+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2} = P(r,\theta).$$

Thus,

$$P(r,\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} = \operatorname{Re} \frac{1+z}{1-z} = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

Using this series form of the Poisson kernel, we get the (complex) Fourier series representation for Φ:

$$\Phi(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it}) P(r, \theta - t)\, \mathrm{d}t =$$

$$\frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it}) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}\, \mathrm{d}t =$$

$$\sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it}) e^{-int}\, \mathrm{d}t \right) r^{|n|} e^{in\theta} = \sum_{n=-\infty}^{\infty} c_n\, r^{|n|} e^{in\theta}.$$

Since Φ is real, we can write $c_n = \alpha_n + i\beta_n$ where

$$\alpha_n = \frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it}) \cos nt\, \mathrm{d}t \quad (\alpha_{-n} = \alpha_n),$$

$$\beta_n = -\frac{1}{2\pi}\int_0^{2\pi} \Phi(e^{it}) \sin nt\, \mathrm{d}t \quad (\beta_{-n} = -\beta_n).$$

This gives $\displaystyle\sum_{n=-\infty}^{\infty} r^{|n|}(\alpha_n + i\beta_n)(\cos n\theta + i \sin n\theta) =$

$$\sum_{n=-\infty}^{\infty} r^{|n|}(\alpha_n \cos n\theta - \beta_n \sin n\theta)$$

$$+i \sum_{n=-\infty}^{\infty} r^{|n|}(\alpha_n \sin n\theta + \beta_n \cos n\theta).$$

Using $\alpha_{-n} = \alpha_n$ and $\beta_{-n} = -\beta_n$, the second sum vanishes and the first becomes the ordinary Fourier series representation of Φ: $\alpha_0 + \displaystyle\sum_{n=1}^{\infty} 2\alpha_n r^n \cos n\theta - 2\beta_n r^n \sin n\theta$.

Here $\alpha_0 = \dfrac{1}{2\pi}\displaystyle\int_0^2 \pi\Phi(e^{it})\,\mathrm{d}t$, $\alpha_n = \dfrac{1}{\pi}\displaystyle\int_0^2 \pi\Phi(e^{it}) \cos nt\,\mathrm{d}t$, $\beta_n = \dfrac{1}{\pi}\displaystyle\int_0^2 \pi\Phi(e^{it}) \sin nt\,\mathrm{d}t$, for $n \neq 0$.

Example. Find $\Phi(re^{i\theta})$ if $\Phi(e^{i\theta}) = \cos^2\theta$.

As $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, we have

$$\Phi(re^{i\theta}) = \frac{1}{2} + \frac{r^2}{2}\cos 2\theta.$$

Example. Find $\Phi(re^{i\theta})$ if $\Phi(e^{i\theta}) = \sin\theta\cos\theta$.

As $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, we have

$$\Phi(re^{i\theta}) = \frac{r^2}{2}\sin 2\theta.$$

Mean value property. The Poisson formula for a harmonic function Φ in a disk $|z| < R$:

$$\Phi(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(Re^{it}) \frac{R^2 - r^2}{R^2 - 2Rr\cos(t - \theta) + r^2} \, dt.$$

For $r = 0$, we get $\Phi(0) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(Re^{it}) \, dt$.

This implies the **mean value property** of a harmonic function Φ in a domain D:

For any circle C in D centered at $z_0 \in D$, the value $\Phi(z_0)$ equals the mean value of Φ on C.

Upper and lower bounds for the Poisson kernel.
As $(R-r)^2 \le R^2 - 2Rr\cos\theta + r^2 \le (R+r)^2$, we have

$$\frac{R^2-r^2}{(R+r)^2} \le \frac{R^2-r^2}{R^2-2Rr\cos(\theta)+r^2} \le \frac{R^2-r^2}{(R-r)^2}.$$

As $\dfrac{R^2-r^2}{(R+r)^2} = \dfrac{R-r}{R+r}$ and $\dfrac{R^2-r^2}{(R-r)^2} = \dfrac{R+r}{R-r}$, this implies

$$\frac{R-r}{R+r} \le \frac{R^2-r^2}{R^2-2Rr\cos(\theta)+r^2} \le \frac{R+r}{R-r}.$$

Combining this with the Poisson formula and mean value property, we obtain **Harnack's inequality** for a positive harmonic function Φ in $|z| \le R$:
If $|z| = r < R$ then

$$\frac{R-r}{R+r}\Phi(0) \le \Phi(z) \le \frac{R+r}{R-r}\Phi(0).$$

The **maximum principle** says that a function Φ harmonic in a domain D cannot take its maximal value inside D, unless it is a constant. Replacing Φ by $-\Phi$, we obtain the **minimum principle**.

Proof. If Φ takes its maximal value M at $z_0 \in D$ then, from the mean value property, it must be a constant (equal M) in any disk centered at z_0 and contained in D. Since D is connected, we can repeat this argument for any point of that disk, eventually covering a path from z_0 to any other point of D by disks such that $\Phi \equiv M$ inside each of them.

A similar argument with Cauchy's formula shows that $|f(z)|$ has a maximum principle for an analytic function $f(z)$. However, $|f(z)|$ does not have a minimum principle unless $f \neq 0$ in D, then $|f(z)|$ is harmonic.