Moduli Lecture 1

Nicolas Diaz-Wahl

1 Motivation

1.1 Moduli of circles

The moduli space of circles with a fixed origin is $(0, \infty)$; circle determined by its radius.

If it's circles in the plane, the moduli is $\mathbb{R}^2 \times (0, \infty)$.

Neither is compact; consider circles tangent to a given point; moduli is $S^1 \times (0, \infty)$. To compactify, add "degenerate" circles, namely a point and "infinite circle" (a straight line). Then the moduli space is $S^1 \times [0, \infty] \simeq S^1 \times [0, 1]$ is a cylinder.

1.2 Inscribed Rectangles

Let C be a Jordan curve. It is a remarkable fact that every such curve contains an inscribed rectangle. It is not known, but strongly suspected, that every closed curve contains an inscribed square. The proof uses moduli-theoretic ideas.

A rectangle is equivalent to two equal length line segments with a common midpoint. Thus the idea is to parametrize pairs of points on C together with their midpoint. For each pair $(x, y) \in C^2$, let m be their midpoint, and $(m, h) \in \mathbb{R}^2 \times \mathbb{R} \simeq \mathbb{R}^3$ be the point at height $h = |x - y|$ above m. The set of all such points defines a surface $M \subset \mathbb{R}^3$; thus two equal distance line segments intersect at a common midpoint if and only if M has a self-intersection! Careful consideration shows that M is a Möbius strip whose boundary lies on C. If B is the interior of C, then $B \cup M$ is a disk and a Möbius strip glued along their boundaries, i.e. a real projective plane. But if M has no self-intersection, $M \cup B$ is a real projective plane embedded in $\mathbb{R}^3$, which is impossible!

2 Moduli in algebraic geometry

The two examples above are for motivation only, as they are not algebro-geometric (in topology, they are usually called *classifying spaces*). Moduli spaces also show up in differential geoemtry (moduli of flat connections, Hitchin moduli spaces, etc.) Now we give some examples of moduli spaces in algebraic geometry.

2.1 Review of Algebraic Varieties

The classical definition of an affine (resp. projective) algebraic variety over a field k is the set of solutions to a system of (homogeneous) polynomial equations $f_1, \ldots, f_r$ in k^n (resp. $k\mathbb{P}^n := (k^{n+1} - \{0\})/k^\times$). A quasi-projective variety is an open subset of a projective variety (using Zariski topology). This includes affines. Sometimes people require varieties to be irreducible.

Consider the variety $x^n + y^n = 1$ in $\mathbb{Q}^2$ for $n \geq 2$. When $n = 2$, this is a dense subset of a circle. When $n \geq 3$ however, this only has $(\pm 1, 0)$ or $(0, \pm 1)$ as solutions. But over $\mathbb{C}$, this is a beautiful curve. We want to "see" this beautiful curve even if it's only defined over $\mathbb{Q}$. For this we need schemes. This brings us to our true definition of affine and projective varieties.

Definition 1. We define *affine n-space* (resp. *projective n-space*) over $\mathbb{Z}$ as the schemes $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[T_1, \ldots, T_n]$ (resp. $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[T_0, \ldots, T_n]$). Their relative versions over an arbitrary scheme S are given by $\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$ and $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$.

An *affine* (resp *projective*) *variety* over a field k is a (reduced) closed k-subscheme of $\mathbb{A}^n_k$ (resp. $\mathbb{P}^n_k$).

An *algebraic variety* over a field k is a separated k-scheme of finite type.

Examples: $\operatorname{Spec} k[x,y]/(y^2 - 4x^3 - g_2 x - g_3)$; $\operatorname{Spec} k[x]/(x^n)$; $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2 + y^2 \pm 1)$; $\operatorname{Spec} \mathbb{R}[x,y]/(x^2 + y^2)$.

Definition 2. A variety $X \to \operatorname{Spec} k$ is *geometrically reduced (connected, irreducible, integral)* if $X_{\bar{k}}$ has the corresponding property. Read Qing Liu Section 3.2.

Varieties often naturally come in families. What this means is that given any surjective morphism of schemes $X \to S$, we can think of the base as "parametrizing" a family of varieties by looking at fibres over points: $X_s \to \operatorname{Spec} k(s)$. Here's a nice example:

Example 1. Let $E \to \operatorname{Spec} \mathbb{Q}$ be an elliptic curve defined by the Weierstrass equation $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$. For this equation to define a smooth curve, the RHS must have no repeated roots, i.e. the discriminant $4g_2^3 - 27g_3^2$ must be nonzero. Suppose that the discriminant is an integer (which can be achieved after a suitable linear change of variables). Then the primes dividing the discriminant are those for which E has "bad reduction". What does this mean? An *integral model* for E is a scheme $\mathcal{E} \to \operatorname{Spec} \mathbb{Z}$ whose generic fibre is E. The special fibres $\mathcal{E}_{\mathbb{F}_p} \to \operatorname{Spec} \mathbb{F}_p$ are the reductions of the elliptic curve (take the equation and reduce mod p), and those with nonzero discriminant mod p have E_p smooth. Thus $\mathcal{E} \to \operatorname{Spec} \mathbb{Z}[1/\Delta]$ where Δ is the discrimiant is a family of elliptic curves.

For another example, consider $\mathcal{E} \to \operatorname{Spec} \mathbb{C}[t]$ via $E : y^2 = x(x-1)(x-t)$. Specializing along various values of t yields a family of elliptic curves. In fact *all* elliptic curves over $\mathbb{C}$ arise in this way, so in some sense $\mathcal{E}$ is the *universal* elliptic curve over $\mathbb{C}$.

A morphism of schemes $X \to S$ is called a *universal fibre space* if $\mathcal{O}_S \to f_* \mathcal{O}_X$ is an isomorphism after arbitrary base change.

Proposition 3. *A proper flat morphism with geometrically reduced and connected fibres is a universal fibre space.*

Proof. Since all properties are stable under base change, we are free to stay at the ground scheme. By proper flat base change, it is enough to show that $H^0(X_s, \mathcal{O}_{X_s}) = k(s)$. Since $X_s \to \operatorname{Spec} k(s)$ is proper, $H^0(X_s, \mathcal{O}_{X_s})$ is finite-dimensional over $k(s)$ (Qing Liu Cor. 3.3.19). Since X_s is reduced, $H^0(X_s, \mathcal{O}_{X_s})$ is reduced over $k(s)$; since X_s is connected, $H^0(X_s, \mathcal{O}_{X_s})$ has a unique minimal prime ideal. Hence $H^0(X_s, \mathcal{O}_{X_s})$ is a finite-dimensional reduced integral domain over $k(s)$, hence algebraic over $k(s)$. Since these properties are stable under base change, and $H^0((X_s)_K, \mathcal{O}_{X_K}) \simeq H^0(X_s, \mathcal{O}_{X_s}) \times_{k(s)} K$ by flat base change (you can also argue as Qing Liu does in Cor. 3.3.21), we must have equality. □

2.2 Projective space as a moduli space

What is the relationship between k-valued solutions to polynomial equations and S-valued points on schemes as given above?

Proposition 4. *Let $X \to \operatorname{Spec} k$ be a (quasi-)projective variety, i.e.*

$$X = V_+(f_1, \dots, f_r) \cap U \subset \mathbb{P}^n.$$

Then

$$X(k) := \operatorname{Mor}(\operatorname{Spec} k, X) \xrightarrow{\sim} \{x \in U \subset \mathbb{P}^n(k) \;:\; f_i(x) = 0 \; \forall i = 1, \dots, r\}.$$

Proof. Given $x \in Z_+(f_1, \dots, f_r)$ and let $\mathfrak{m}_x = (T_0 - x_0, \dots, T_n - x_n)$, a maximal ideal in X ($I \subset \mathfrak{m}_x$) with residue field k; explicitly,

$$\mathcal{O}_X(D_+(T_i))/\mathfrak{m}_x \to k : f(t) \mapsto f(x) \bmod \mathfrak{m}_x.$$

Conversely, given $x \in X(k)$, set x_i to be the image of T_i in $k(x) = k$; then $f(x) = 0$ for all $f \in I$ (look at Taylor expansion about x) and we win. □

So this hints at the importance of sections of the structure map $X \to \operatorname{Spec} k$. What about R-algebras? S-schemes? Say we want a definition of $R\mathbb{P}^n$, i.e. R-lines in R^{n+1}. But if you think carefully, if R is not a domain, and a is a zero-divisor, aR is not a free R-module, so should it count as a subspace? Perhaps only the 1-dimensional free submodules should count?

If we want lines up to *isomorphism*, we need to distinguish $\mathbb{Z}(1,1) \subset \mathbb{Z}^2$ from $\mathbb{Z}(2,2)$, so we should only consider $\mathbb{Z}(a,b)$ with $\gcd(a,b) = 1$. Scheme theoretically, we want $\mathbb{P}^1(X)$ to *line bundles generated by a two global sections*. Similarly, $\mathbb{P}^2(\mathbb{Z})$ should be the set of $\mathbb{Z}(a,b,c)$ with $\gcd(a,b,c) = 1$, i.e. line bundles generated by three global sections. So $\mathbb{P}^n(R)$ should be line bundles generated by $n+1$-global sections (i.e. elements of R). This extends to arbitrary schemes.

Upgrading to $X \to S$, the S-valued points are $X(S) = \{S \to X\}$ sections to $X \to S$.

Theorem 5 (Qing Liu Prop 5.1.31)**.** *Let S be any scheme, $X \to S$ an S-scheme, and $\mathbb{P}^n_S$ projective n-space over S (defined as $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} S$). Then*

$$\mathbb{P}^n_S(X) = \{(\mathcal{L}, s_0, \dots, s_n) \;:\; s_i \text{ global generators of } \mathcal{L}\}.$$

Proof. Fix a map $f : X \to \mathbb{P}^n_S$. Since $\mathcal{O}_{\mathbb{P}^n_S}(1)$ is a line bundle generated by the global sections T_i, we obtain the line bundle $\mathcal{L} := f^*\mathcal{O}_{\mathbb{P}^n_S}(1)$ and global generators $s_i := f^*T_i$ of $\mathcal{L}$. Then pass to isomorphism classes. Conversely given $\mathcal{L}$ and global generators $s_0, \dot{,} s_n$, the open sets $X_{s_i} = \{x \in X \;:\; (s_i)_x\mathcal{L}_x = \mathcal{L}_x\}$ cover X, so the morphisms $f_i : X_{s_i} \to D_+(T_i)$ corresponding to ring maps

$$\mathcal{O}_{\mathbb{P}^n_S}(D_+(T_i)) \to \mathcal{O}_X(X_{s_i}) : T_j/T_i \mapsto s_j/s_i$$

glue to a map $f : X \to \mathbb{P}^n_S$ satisfying $f^*\mathcal{O}(1) = \mathcal{L}$. □

Notice how promoting $\mathbb{P}^n$ to a scheme-theoretic definition yields a *functorial* interpretation of projective space. Thus $\mathbb{P}^n$ is a "moduli space of globally generated line bundles together with their generators". Let's introduce some basic category theory:

Definition 6. Let $\mathcal{C}$ be a category; a functor $F : \mathcal{C}^o \to \text{Set}$ is called a (set-valued) *presheaf*. We say F is *representable* by an object $X \in \mathcal{C}$ if there is a natural transformation $\eta : F \simeq h_X$ where $h_X : \mathcal{C}^0 \to \text{Set}$ is the functor $h_X(T) = \text{Hom}(T, X)$ and $h_X(T' \xrightarrow{f} T) : \text{Hom}(T, X) \to \text{Hom}(T', X) : \varphi \mapsto \varphi \circ f$.

Proposition 7 (Yoneda's lemma)**.** *Let $\mathcal{C}$ be any category, and X be an object of $\mathcal{C}$. The functor $\mathcal{C} \to \text{PSh}(\mathcal{C}) : X \mapsto h_X$ is fully faithful. Moreover, if $F \in \text{Psh}(\mathcal{C})$, then the functor $\text{Func}(\mathcal{C}, \text{PSh}(\mathcal{C})) \to \text{PSh}(\mathcal{C} \times \text{PSh}(\mathcal{C})) : (F\, \eta\, h_X) \mapsto FX$ is a natural transformation.*

Proof. Exercise. □

A representable presheaf $F \simeq h_X$ comes with a *universal object* $\xi \in FX$ corresponding to the identity morphism $\text{id}_X : X \to X$. Analyzing the map in Yoneda's lemma, we find that any object $\theta \in FT$ is given by the pullback $f^*\xi$ where $f : T \to X \in \text{Hom}(T, X)$ is the map corresponding to θ.

Note also how this implies that $\text{Hom}(X, Y) \simeq \text{Hom}(h_X, h_Y)$. This is Grothendieck's *relative point of view*; instead of studying an object X, study all maps $T \to X$. We can now define moduli spaces for real:

Definition 8. Let $F : (\text{Sch}\,/S)^{\text{op}} \to \text{Set}$ be a functor. This is called a *moduli problem.* A *fine moduli space* for F is a scheme X together with a natural transformation $F \simeq h_X$.

As seen in the theorem, the pair $(\mathbb{P}^n_S, (\mathcal{O}(1), T_0, \ldots, T_n))$ is the fine moduli space representing the functor $X \mapsto \{(\mathcal{L}, s_0, \ldots, s_n)\}/\sim$.

2.3 Touch grass Grassmannians and Quot Schemes

(Reference: Lee *Smooth Manifolds* Example 1.36) Let V be a vector space over a field k of finite-dimension n. Classically, Grassmannian varieties/manifolds are given by the following data: as a set, $\text{Gr}_p(V) = \{p\text{-dimensional subspaces of } V\}$; if $k = \mathbb{R}$ or $\mathbb{C}$; then $\text{Gr}_p(V)$ acquires the structure of a $p(n-p)$-dimensional smooth (resp. analytic) manifold. Fix complementary subspaces P, Q of dimensions p, $n-p$ so that $V = P \oplus Q$. The graph a linear map $T : P \to Q$ may be identified with a p-dimensional subspace of V, specifically $\Gamma(T) = \{v + Tv \,:\, v \in P\}$ which meets Q trivially. Conversely, any complementary subspace S to Q is the graph of a unique linear map $T : P \to Q$, specifically $(\pi_Q|_S) \circ (\pi_P|_S)^{-1}$.

The subset U_Q of $\text{Gr}_p(V)$ of p-dimensional subspaces meeting Q trivially and the space of linear maps $\text{Lin}(P, Q)$ are identified via $T \mapsto \Gamma(T)$, so $U_Q \simeq k^{p(n-p)}$. One can check that the transition functions involve only products and inverses of matrices, and are thus polynomials. This invites a scheme-theoretic, and even moduli-theoretic interpretation of $\text{Gr}_p(V)$.

Just like with projective space, we defined for instance $\mathbb{P}^n(R)$ as line bundles on $\text{Spec}\, R$ generated by $(n+1)$-global sections. Since we are now parametrizing *p-dimensional* subspaces as opposed to 1-dimensional, line bundles should be replaced by locally free sheaves.

If you think carefully about the manifold construction, one notes that the "points" of $\text{Gr}_p(V)$ are complements of an $n-p$-dimensional subspace, which may be thought of as kernels of projection mappings $\pi_Q : V \to Q$. The difference is that now we take the "dual" definition, i.e. projections onto p-dimensional subspaces. This is what we use to define the scheme-theoretic Grassmannian.

One might define the Grassmann functor as $\underline{Gr}_p : (\mathrm{Sch}/S)^o \to \mathrm{Set} : S \mapsto \{\pi : \mathcal{O}_S^{\oplus n} \to V\}/\sim$ where $\mathcal{V}$ is p-dimensional. If $\mathcal{E} = \ker \pi$, then $0 \to \mathcal{E} \to \mathcal{O}_S^{\oplus n} \to \mathcal{V} \to 0$ is exact, so on fibres we have $0 \to \mathcal{E}_{k(s)} \to k(s)^{\oplus n} \to \mathcal{V}_{k(s)} \to 0$ is exact, so $\mathcal{E}_s$ is a rank $n-p$-dimensional subspace, so the inclusion $\mathcal{E} \to \mathcal{O}_S \otimes V$ is a family of rank $n-p$ subspaces of an n-dimensional vector space parametrized by S, i.e. rank $n-p$ subbundles of $\mathcal{O}_S^{\oplus n}$.

Theorem 9. $\mathrm{Gr}(p,n)$ *is representable by a smooth projective scheme of dimension* $p(n-p)$ *over* $\mathrm{Spec}\,\mathbb{Z}$.

In fact, we will define a slightly more general functors and prove their representability. First, we will have the "relative Grassmannians", $\mathrm{Gr}_S(p, \mathcal{E})$ whose $T \xrightarrow{f} S$-valued points are isomorphism classes of surjections $f^*\mathcal{E} \to \mathcal{V}$ where $\mathcal{V}$ is a rank k locally free sheaf. Their representability by a smooth projective S-scheme is proved by reducing to the case of a usual Grassmannian by an easy glueing argument.

2.4 We touched grass, now what?

The next goal will be to define the Hilbert and Quot schemes. Instead of parametrizing linear subspaces, we want to parametrize *arbitrary* (algebraic) subspaces. For these to fit nicely in families, they must be flat over the base scheme. So we obtain a functor

$$\underline{\mathrm{Hilb}}_{X/S} : T \mapsto \{\text{Closed subschemes } Z \subset X_T \ : \ Z \to T \text{ is flat and proper}\}.$$

While the properness assumption seems to be more restrictive than projective space (since lines in $\mathbb{A}^n$ are not proper), projective space is equivalent to lines in $\mathbb{P}^n$ (a circular definition of course, but useful in hindsight!). To make sure that the resulting scheme isn't "too big" (in particular, not quasi-compact), we will need to restrict the types of by their Hilbert polynomials. We will see this in a couple of weeks.

Hilbert schemes are crucial, as they allow us to define many more moduli spaces. For instance, the *relative divisor scheme* $\mathrm{Div}_{X/S}$ is an open subscheme of $\mathrm{Hilb}_{X/S}$ given by $\mathrm{Div}_{X/S}(T) = \{\text{relative effective Cartier divisors on } X_T\}$. We also have the *Picard scheme*, which it is the goal of this course to construct. It is given as follows: $\mathrm{Pic}_{X/S}(T) = \{\text{Line bundles on } X_T \text{ modulo pull backs from } \mathrm{Pic}(T)\}$. Let's quickly give a spectacular application of the existence of $\mathrm{Pic}_{X/S}$.

Theorem 10 (Torelli)**.** *The functor* $C \mapsto (\mathrm{Pic}^0_{C/k}, \theta_C)$ *from the category of (smooth, geometrically integral) curves of genus* $g > 1$ *to principally polarized abelian varieties is fully faithful. It's almost fully faithful when* $g = 1$.

If A an abelian variety, then $\mathrm{Pic}^0_{A/k}$ is the *dual abelian variety* of A, and plays the role of the dual vector space in linear algebra. It's importance cannot be overstated.

3 Moduli of curves and abelian varieties

Finally, we mention the most well-known and widely used applications of moduli spaces (at least to me), that is the moduli of curves and abelian varieties. Let M_g be the functor mapping a scheme S to the set of curves of genus g over S. This is representable by a *smooth Deligne-Mumford stack* when $g > 1$, and cool things are true for $g = 0, 1$ as well. Talk about $M_{g,n}$ and semistable compactification.

The functor $M_{1,1}$ is the *moduli space of elliptic curves* or *modular stack*. If we take pairs (E, P) where P is a point of order N on E, then the moduli of such pairs is denoted $Y_1(N)$ and is called a *modular curve of level* $\Gamma_1(N)$. It is a smooth scheme over $\mathbb{Z}[1/N]$ when $N \geq 3$, and admits a nice compactification $X_1(N)$ using semistable reduction theory. This is also the case for M_g. This brings us to another spectacular application of the existence of $X_1(N)$ and Picard schemes!

Theorem 11 (Eichler-Shimura)**.** *Let $N \geq 3$, and p be a prime not dividing N. Set*

$$J_p = \mathrm{Pic}^0_{X_1(N)/\mathbb{F}_p} = \mathrm{Pic}^0_{X_1(N)/\mathbb{Z}[1/N]} \times_{\mathbb{Z}[1/N]} \mathbb{F}_p.$$

Let the Hecke algebra $T_1(N)$ act through the lower star action. Then in $\mathrm{End}_{\overline{\mathbb{F}}_p}(J_p)$*, we have*

$$(T_p)_* = F + \langle p \rangle_* F^\vee, \quad w_\zeta^{-1} F w_\zeta = \langle p \rangle_*^{-1} F.$$

There are also stacks of abelian varieties. An abelian variety has a dimension (call it g), a *polarization* which is a preferred map $A \to \mathrm{Pic}^0_{A/k}$, just like the elliptic curve case, it can be given a *level structure* (point of order N), and its endomorphism ring is very interesting (for elliptic curves, we have ordinary, CM, and over finite fields, supersingular, corresponding to $\mathbb{Z}$, orders in imaginary quadratic fields, and quaterion algebras over $\mathbb{Z}$). Their moduli are denoted $\mathcal{A}_g$, $\mathcal{A}_{g,N}$, $\mathcal{A}_{g,N,d}$ and $\mathrm{Sh}(G, X)$. The latter is called a PEL-Shimura variety (polarization, endomorphism, level structure). The $\mathcal{A}_{g,N,d}$ are critical in Falting's proof of Mordell's conjecture. Shimura varieties are the centerpiece of the Langlands program.