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The Index of Discontinuous Vector Fields

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Abstract. The concept of the index of a vector field is one of the oldest in Algebraic Topology. First stated by Poincare and then perfected by Heinz Hopf and S. Lefschetz and Marston Morse, it is developed as the sum of local indices of the zeros of the vector field, using the idea of degree of a map and initially isolated zeros. The vector field must be defined everywhere and be continuous. A key property of the index is that it is invariant under proper homotopies.

In this paper we extend this classical index to vector fields which are not required to be continuous and are not necessarily defined everywhere. In this more general situation, proper homotopy corresponds to a new concept which we call proper otopy. Not only is the index invariant under proper otopy, but the index classifies the proper otopy classes. Thus two vector fields are properly otopic if and only if they have the same index. This allows us to go back to the continuous case and classify globally defined continuous vector fields up to proper homotopy classes. The concept of otopy and the classification theorems allow us to define the index for space-like vector fields on Lorentzian space-time where it becomes an invariant of general relativity.

Contents

Introduction

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A. The results. We generalize the notion of homotopy of vector fields to that of otopy of vector fields. Using otopy we can:

1. Classify the proper homotopy classes of vector fields. The index is a proper homotopy class invariant but two vector fields with the same index may not be proper homotopic. (See (4) in Section 5.)

2. Show that two vector fields are properly otopic if and only if they have the same index. (See (3).)

3. Extend the definition of index to any vector field, without hypotheses. (See Subsection C in the Introduction.)

4. Define a local index for any connected set of defects for any vector field instead of merely for isolated zeros. (See the penultimate paragraph of C or Section 4.) Under an otopy these defects move and interact. The following conservation law holds: The sum of the indices of the incoming defects is equal to the sum of the indices of the outgoing defects. (See (1).)

5. Demonstrate that the concept of the index of a vector field depends only on elementary differential topology, the concept of pointing inside, and the Euler-Poincare number. This is done in Sections 1, 2, 3, 4. The classical approach depends on the degree of a map. In this paper we show how the degree of a map might be defined via the index of a vector field by using (11).

6. We can study vector fields along the fibre on fibre bundles. An otopy generalizes to a vector field V along the fibre restricted to an open set. For a proper V, only certain values of the index of V restricted to a fibre are possible. (See (14).) For example, the Hopf fibrations of spheres admit only the index zero.

7. We can study space-like vector fields on a Lorentzian space-time, M. The concept of otopy generalizes to a space-like vector field restricted to an open set of M. For a proper space-like vector field V, the index of V restricted to a space-like slice is independent of the slice. Thus, the index is an invariant of General Relativity. Hence it should be used to describe physical phenomena. For example, the Coulomb electric vector field E of an electron or a proton has index -1 or 1 respectively. This remains true no matter what what coordinate system is used to describe the field.

B. Organization of the paper. There are a few features to be explicitly noted. First, we are actually defining two types of indices. These are usually denoted $\mathrm{Ind}_U(V)$, and $\mathrm{ind}(P)$. The first takes values in the integers and ∞ and the last takes on the value $-\infty$ as well. Second, there are two different definitions of these indices. The advanced definition is based on the definition of index already defined for continuous vector fields and is found in Subsection C of the Introduction. The elementary definition is given inductively in Sections 1, 2, 3, 4. This definition is equivalent to the first, and the proof that it is well-defined is completely self contained, using only pointset topological methods. The only algebraic topological notion is that of the Euler Poincare number.

Subsection C of the Introduction establishes notation and the formal concept of proper otopy as well as the key example of otopy which forces the concept on us. If the reader draws a few pictures and understands what is to be formalized, the formal definitions will be obvious except for a few small details. The formal definition of otopy is in Subsection C along with the advanced definition of index.

Subsection D of the Introduction contains guides for three different ways for reading the paper. Especially contained in D is a simplified description of the index which if combined with the list of key properties of Section 5 should give the reader the essence of the subject without the technicalities of the proof.

The main burden of the paper is the development of the elementary definition of the index. It is here that the two different types of index, $\mathrm{Ind}_U(V)$, and $\mathrm{ind}(P)$, are carefully defined. The definition is made inductively on the dimension of the manifolds and is shown to be well-defined. This takes up Sections 1, 2, 3, and 4.

In Section 5 we write a list of 14 properties of the index. There are short proofs of them. It is hoped that this list will be easy to use for the mathematician or physicist who needs to apply the idea of index in their work.

C. Definition of otopy and index. The concept of otopy arises from homotopy in a natural way. Consider a smooth compact manifold M with boundary ∂M. Let V be a continuous vector field defined on M. There is an associated vector field ∂V on ∂M given by projecting the vectors of V on ∂M to vectors which are tangent to ∂M. (See the paragraph above Lemma 2.1 for the definition of projection.) Denote by $\partial_- M$ the open set of ∂M where the vectors of V point inside. Let $\partial_- V$ denote ∂V restricted to $\partial_- M$. For outward pointing vectors we define $\partial_+ M$ and $\partial_+ V$. Let $\partial_0 M$ denote the closed set of ∂M where the vectors of V are tangent to ∂M.

Now consider a homotopy V_t. It induces a homotopy ∂V_t on ∂M. Now $\partial_- V_t$ is varying with t, but it is not a homotopy. We say it is an *otopy*. It is the key example of an otopy.

THE KEY OBSERVATION ABOUT OTOPIES. Consider a zero of ∂V which passes from $\partial_- M$ to $\partial_+ M$ in ∂M. As it passes over $\partial_0 M$ it coincides with a zero of V which is passing through ∂M.

Thus there is a connection between the zeros of V_t which pass inside and outside of M through ∂M and the zeros of ∂V_t which pass inside and outside of $(\partial_- M)_t$. The concepts of *proper* homotopies and *proper* otopies and *proper* vector fields are introduced so that no zeros appear on ∂M or $\partial_0 M$.

Definition of continuous otopy. Let N be a manifold and let V be a continuous vector field defined on $N \times I$ so that V is tangent to the slices $N \times t$. Then we say that V is a continuous *homotopy* and that $V_0 = V(m, 0)$ and $V_1 = V(m, 1)$ are homotopic vector fields. Suppose that T is an open set on $N \times I$ and V is a continuous vector field defined on T so that V is tangent to the slices $N \times t$. Then we say that V is a continuous *otopy* and that V_0 and V_1 are *otopic*. Note that V_0 or V_1 are vector fields defined only on the open sets in M given by the intersection of T with $M \times i$ for $i = 0$ or 1. Thus V_0 or V_1 can be "empty" vector fields. Also note that "otopy" gives an equivalence relation on the set of vector fields defined on open sets in N. This follows just as in the homotopy case. But it is a trivial equivalence relation, since every vector field is homotopic to the zero vector field and every vector field field defined on an open set is otopic to the empty vector field.

Definition of proper continuous otopy. If U is an open set in a manifold with boundary we will adopt the convention that the *Frontier* of U includes that part of the boundary ∂M of M which lies in U, as well as the usual frontier. The capital F will distinguish these two different notions of Frontier and frontier. We say that V defined on an open set U in N is a *proper* vector field on the domain U if the zeros of V form a compact set in U and if V extends continuously to a vector field on $\overline{U}$ with no zeros on the Frontier of U. Thus if V is defined on a compact manifold M with boundary ∂M, we say V is *proper* if there are no zeros on ∂M. A *proper otopy* with domain T is an otopy V defined on the open set T with a compact set of zeros whose restriction to any slice is a proper vector field. A *proper homotopy* is an proper otopy V defined on all of $N \times I$.

Now the index for proper continuous vector fields on a compact manifold with or without boundary, as well as the index for proper continuous vector fields defined on an open set U, were defined in [BG]. If V is a continuous proper vector field defined on a compact manifold M with boundary ∂M and Euler-Poincare number $\chi(M)$, then $\partial_- V$ is also proper and the index satisfies

$$\mathrm{Ind}V = \chi(M) - \mathrm{Ind}(\partial_- V)$$

Now we consider any arbitrary, possibly discontinuous, vector field V on N. We assume we are in a smooth manifold N. A vector field is an assignment of tangent vectors to some, not necessarily all, of the points of N. We make no assumptions about continuity. We consider the set of defects of a vector field V in N, that is the set D which is the closure of the set of all zeros, discontinuities and undefined points of V. That is we consider a *defect* to be a point of N at which V is either not defined, or is discontinuous, or is the zero vector, or which contains one of those points in every neighborhood.

We extend the notion of *proper* to arbitrary tangent vector fields by replacing the word zero by defect.

Definition of discontinuous proper otopy. We say that V is a *proper* vector field on an open set U if the defects of V in U form a compact set and if V can be extended to $\overline{U}$ so that there are no defects on the Frontier of U. Thus for N a compact manifold with boundary we say that V is a *proper vector field* if there are no defects on the boundary. A *proper otopy* W with domain T is an otopy in $N \times I$ whose defects form a compact set and whose restriction to every slice is a proper vector field for that slice. A globally defined otopy is still called a *homotopy.* We will modify the word homotopy to discontinuous homotopy if needed.

REMARKS. 1. As before, the concept of proper discontinuous otopy is an equivalence relation on the locally defined vector fields of N. It is a simple exercise of pointset topology to show that every discontinuous vector field is otopic to a continuous vector field. Also, if two continuous locally defined vector fields are otopic, they are continuously otopic. So the extension of index theory from continuous to discontinuous vector fields is not mathematically challenging. But discontinuous vector fields arise very naturally in mathematics and physics and now the results of index theory can be applied to them without any mental anguish.

2. In order to avoid confusion between points at which the vector field V is undefined inside the open set U and outside the open set U we can restrict our attention without loss of generality to vector fields which are defined everywhere on U, but

are not necessarily continuous. Any vector field on U which is not defined at some points in U can be replaced by the same vector field returning the zero vector at those undefined places. In fact, for those readers who are uncomfortable with the notion of discontinuous vector fields, Remark 1 offers a way to proceed by thinking of only continuous vector fields.

3. Note that a defect of an otopy need not be a defect of the vector field defined on the slice. For example, consider the unit vector field pointing to the right on the real line and otopy it to the unit vector field pointing to the left by letting the field reverse direction when $t = 1$. The vector field at $t = 1$ has no defects thought of as a vector field on the real line, but the otopy defects are located at all the points of the $t = 1$ slice. Thus this otopy is not proper since the set of defects is not compact. Replacing the line by a closed interval, the above example has a compact set of defects, but it still is not a proper otopy because defects are on the Frontier of the $t = 1$ slice. If we replace the real line by a circle in our example, we again get defects on the top circle, but they form a compact set and there is no Frontier, so we consider this as a proper otopy, indeed a proper discontinuous homotopy.

The advanced definition of index. If V is a proper discontinuous vector field defined on a compact M with boundary, then $\partial_- V$ is continuously defined on $\partial_- M$. So $\mathrm{Ind}(\partial_- V)$ is defined. Then $\mathrm{Ind}(V)$ is defined by $\mathrm{Ind} V = \chi(M) - \mathrm{Ind}(\partial_- V)$. If V is a proper discontinuous vector field "defined" on the open domain U, we can find a compact manifold M which contains the compact set of defects of V and has none on ∂M. Then $\mathrm{Ind}_U V := \mathrm{Ind}_M V$ where $\mathrm{Ind}_M(V)$ means the index of V restricted to M.

If V is not a proper vector field on U , then we define $\mathrm{Ind}_U(V) := \infty$. We introduce ∞ to avoid saying that the index is undefined, since there is information when V is not proper.

Now let P be a connected component of the set of defects D of a vector field V on N. We will define the local index of P, which we denote by $\mathrm{ind}(P)$, as follows: Let U be an open set containing P and no other defects, so that V is proper on U. Then $\mathrm{ind}(P) := \mathrm{Ind}_U(V)$. If there is no such U, but P is contained in an open set on which V is proper, then $\mathrm{ind}(P) := -\infty$. If there is no open set U containing P on which V is proper, then $\mathrm{ind}(P) := \infty$.

The relationship between $\mathrm{Ind}_U(V)$ and $\mathrm{ind}(P)$ is very striking. If all the indices involved are finite, then

$$\mathrm{Ind}_U(V) = \sum \mathrm{ind}(P), \text{ where the sum is over all connected components } P \text{ of } D.$$

D. Guides.

1. The intuitive picture. Consider the defects of a vector field as "topological particles" P_i endowed with a "charge" denoted $\mathrm{ind}(P_i)$. These P_i move and interact as the vector field evolves in time (that is under otopy and homotopy). These "charges" are preserved under collisions just as electric charge is. See (1), the conservation law. Then for a region of space U or M, we have $\mathrm{Ind}_U(V) = \sum \mathrm{ind}(P_i)$ for P_i contained in U, ((8), the summation equation). The list of properties 1–14 in Section 5 can then be used to calculate the index. Particularly useful is the Law of Vector Fields, (2). The classification of otopy by index means that any set of defects P_i can be transformed to any other set of P_j's if and only if the sum of

the indices of the P_i is equal to sum of the indices of the P_j. Note that $\operatorname{ind}(P_i)$ in dimension one can only take on the values, $-1, 0, 1, \infty, -\infty$. In higher dimensions $\operatorname{ind}(P_i)$ can be any integer and ∞ and $-\infty$. The value $-\infty$ will most probably not appear in a physical application.

2. The elementary definition. This is the *only* modern complete account of indices that the authors are aware of. In Section 1 are listed 6 lemmas. The $\operatorname{Ind}_M(V)$ and $\operatorname{Ind}_U(V)$ are assumed to be defined in dimension $n-1$ and satisfy the 6 lemmas. Then for M a compact manifold we define $\operatorname{Ind}_M(V) := \chi(M) - \operatorname{Ind}_{\partial_- M}(\partial_- V)$. In Section 2 we show that this is well-defined. Then for U an open set, we define in Section 3 $\operatorname{Ind}_U(V) := \operatorname{Ind}_M(V)$ where $M \subset U$ contains the defects of V in U. In Section 2 and Section 3 we prove the lemmas in Section 1 for dimension n. The most subtle property to prove is the "existence of defects" (7). The lemmas for dimension 1 are proved in Section 1. After $\operatorname{Ind}_M(V)$ and $\operatorname{Ind}_U(V)$ are established, we define in Section 4 the local index for a "topological particle", that is a connected component P of the set of defects of V, then $\operatorname{ind}(P) := \operatorname{Ind}_U(V)$ where U is an open set containing P and no other defects. The two cases where such a U cannot exist are given by $\operatorname{ind}(P) = \pm\infty$. In the rest of the paper Ind_U and Ind_M will frequently be shortened to Ind.

The prerequisites for this development of index are elementary topology and differential topology. The only "sophisticated" results used are: The Tietze Extension Theorem; the existence of triangulations for smooth manifolds; transversality; smooth approximation to continuous cross-sections; the additivity of the Euler-Poincare number. Most of these can be found in [GP].

In Section 5 all the key properties of the index are listed. Properties (1) to (8) are basically proved in the earlier sections. Properties (9) and (10), the product and sign rules, are proved as simple consequences of properties (1) to (8). Property (11) requires knowledge of the degree of a map. It is this result which shows that the index defined this way agrees with the other definitions as in [BG] or [M]. It should be mentioned that property (11) could stand as a definition of the degree, and presumably most of the properties of degree could be proved from properties (1) to (11). The main point is this: The index is independent of degree, and also intersection number, fixed point index, and coincidence number. Properties (12), (13), (14) are proved elsewhere. The proofs employ the previous properties and sophisticated algebraic topology. Each one is a generalization of a famous theorem.

3. The advanced definition. For the Expert who knows homotopy theory and differential topology well, [BG] will be accessible. The concept of otopy was introduced in that paper, and the invariance of index under otopy was established. (Although otopy was first published there, its actual discovery came from the underlying motivation of this paper: To define the index by means of the Law of Vector Fields.) One should read the definition in Subsection C in the Introduction to extend the definition of the index for discontinuous vector fields. Then to prove the classification theorems, (3) and (4), use the properties of [BG] where needed and the otopy extension property, which is proved in Section 2.

4. An advantage for the elementary definition. The elementary definition is based directly on the vector field, unlike the other definitions. In [BG] a map is constructed and the degree of the map is the index. In Hopf's definition the vector field must be deformed until there are only a finite set of zeros. G. Samaranayake

makes use of this advantage in her thesis [S]. She has a computer program which estimates the index of a zero using the Law of Vector Fields, (2). It works well because she does not need to prepare the vector field in any substantial way. Using this program she can search for zeros of a static coulomb electric field generated with a finite number of electrons and protons whose index is not -1, 0, or 1. Placing protons at the vertices of the Platonic solids: tetrahedron, octahedron, cube, icosahedron, and dodecahedron, she estimates the index of the central zero to be –3, –5, 5, –11, and 11 respectively.

1. The Definition for One-dimensional Manifolds

First we describe the organization of the definition of index and the way we will show it is well-defined. In most situations M will denote a compact smooth manifold with boundary ∂M which is possibly empty. N will usually denote an arbitrary smooth manifold with or without boundary, and U will denote an open set in N or M. We usually consider vector fields V as globally defined over M or locally defined over N. If V is locally defined it is associated to an open set U in N on which it is globally defined. We say U is the *domain* of V.

The inductive definition of index. Let ϕ denote the empty domain. Define $\mathrm{Ind}_\phi(V) := 0$. If M is a compact connected manifold and V is globally defined on M with no zeros on ∂M, then define $\mathrm{Ind}_M(V)$ by

$$(*) \qquad \mathrm{Ind}_M(V) := \chi(M) - \mathrm{Ind}_U(\partial_- V) \text{ where } U = \partial_- M.$$

Let M be a smooth manifold with a globally defined V. Then $\mathrm{Ind}_M(V) :=$ sum of indices on each path component. Let V be a proper vector field on the open set U in N. Define $\mathrm{Ind}_U(V) := \mathrm{Ind}_M(V)$ where M is a compact manifold in U containing the defects of V.

REMARK. It will be clear that by Lemma 1.6, the equation $(*)$ will hold for non-connected manifolds also. We shall refer below to $(*)$ without the connectedness hypothesis.

We begin the induction at dimension -1, the empty manifold. Here the index is zero. For dimension 0, the connected manifold is a point and the vector field V consists of the zero vector. Applying $(*)$ we see that $\mathrm{Ind}_M(V)$ equals 1. Thus $\mathrm{Ind}_U(V)$ equals the number of points in U.

In dimension 1 there are two compact connected manifolds: The circle and the closed interval. Let V be a vector field globally defined on a circle M. Then $\mathrm{Ind}_M(V) = 0$ follows from $(*)$. Note that if V were the zero vector field, it is proper when N is a circle. This contrasts to the fact that a zero vector field can never be proper on an M or a U with non empty Frontier.

Let M be a closed interval and let V be a proper vector field. Then $\mathrm{Ind}_M(V) = 1-$(number of points on the boundary where V points inside). Thus $\mathrm{Ind}_M(V)$ can take on the values 1, 0, -1.

Let M be a general compact 1-dimensional manifold with a globally defined V. Then M is a finite union of closed intervals and circles and $\mathrm{Ind}_M(V) :=$ sum of indices on each path component.

So we have a definition for $\mathrm{Ind}_M(V)$ which is obviously well-defined in one dimension. It will be necessary, however, to prove that $\mathrm{Ind}_M(V)$ is well-defined beginning with dimension 2 for each step of the induction. We must show in dimension 1 already that Ind_U is well-defined. We prove three lemmas about the M case and then after Lemma 1.3 we can show that $\mathrm{Ind}_U(V)$ is well-defined. We state the lemmas in this Section for general manifolds. The proofs will be for dimension one. Frequently Ind will stand for either Ind_M or Ind_U.

Lemma 1.1. *Two vector fields V and V' globally defined on M are properly homotopic if and only if*

$$\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- V') \text{ on each component of the boundary.}$$

Proof. We may assume that M is connected. If M is a circle, every globally defined V is properly otopic to any other globally defined V'. On the other hand, there is no path component of the circle's empty boundary. So the result is trivially true for the circle. Next assume that M is a closed interval. Let W be a vector field so that $W(m) = V(m)/\|V(m)\|$ for m on the boundary of M. Assume that $W(m) = 0$ outside a collar of the boundary, and assume that W continuously decreases in size from the unit vectors on the boundary to the zero vectors at the other end of the collar. Then we define the homotopy $tV + (1-t)W$. This is a proper homotopy, since at any point m on the boundary $V(m)$ and $W(m)$ both point either inside or outside so no zero can arise on the boundary. Now both V and V' are properly homotopic to W, hence they are properly homotopic to each other. □

REMARK. Note that if V and V' are continuous vector fields, there is a continuous proper homotopy between them. If they are smooth, then there is a smooth proper homotopy between them. Also note that for a vector field V globally defined on an interval, there are only four proper homotopy classes. In higher dimensions there are infinitely many proper homotopy classes. The corresponding result in higher dimensions is Theorem 2.2.

Lemma 1.2. *If M is a compact manifold diffeomorphic to M' and the vector field related to V by the diffeomorphism f is denoted by V^*, then*

$$\mathrm{Ind}_M(V) = \mathrm{Ind}_{M'}(V^*)$$

Proof. Pointing inside is preserved under diffeomorphism. □

Lemma 1.3. *If V has no defects, then* $\mathrm{Ind}(V) = 0$.

Proof. We may assume that M is connected. Let M be an interval. Since V has no defects on this interval, V must point outside on one end and inside on the other. Thus $\mathrm{Ind}(V) = 1 - 1 = 0$ on this interval. For M a circle the globally defined V must always have index zero. □

Now we can show that $\mathrm{Ind}_U(V)$ is well-defined. If M and M' are two compact manifolds containing the defects, and contained in U, there is a compact manifold M'' also contained in U and containing both M and M'. The vector field V restricted to $M'' - \mathrm{int}(M)$ is a nowhere zero vector field, and the previous lemma and the fact that the index is additive proves that $\mathrm{Ind}_U(V)$ is well-defined for those vector fields for which the defects sit inside a compact manifold with boundary.

Lemma 1.4. *Given a connected N, two proper locally defined (continuous) vector fields are properly otopic (by a continuous otopy) if and only if they have the same index. For every integer n there is a vector field whose index equals that integer (provided N has positive dimension).*

Proof. Suppose we have a proper otopy W with domain T on $N \times I$. Let V_t denote W restricted to $N \times t$. We show that there is some interval about t such that V_s has the same index for all s in the interval. Since the set of defects of the otopy is compact we can find a compact manifold M so that $M \times J$, for some closed interval J, lies in T and contains the defects inside $\partial M \times J$. Thus the proper homotopy V_t on $M \times J$ preserves the index on M, and hence the proper otopy on $N \times J$ preserves the index on N as t runs over J. Thus we have a finite sequence of vector fields each having the same index as the previous vector field. Hence the first and last vector fields have equal indices.

Conversely, for any integer n, let W_n be the vector field consisting of $|n|$ vector fields defined on disjoint open intervals in N, each one of index 1 if $n > 0$ and of index -1 if $n < 0$. Thus $\mathrm{Ind}(W_n) = n$. Now if V has index n, we must show that V is properly homotopic to W_n. Now the domain of V consists of open connected intervals, and only a finite number of them contain defects. Each of these intervals has index equal to 1, -1, or 0. Now V is properly otopic to the same vector field V whose domain is restricted to only those intervals which have nonzero indices. Now if two adjacent intervals have different indices, there is a proper otopy which leaves the rest of the vector field fixed, and removes the two intervals of opposite indices. After a finite number of steps we are left with either an empty vector field, if $n = 0$, or a W_n. The empty vector field is W_0. Thus V is properly otopic to W_n. □

Lemma 1.5. $\mathrm{Ind}_U(V)$ *on N is invariant under diffeomorphism.*

Proof. Immediate from Lemma 1.2 and the definition of index for locally defined vector fields. □

Lemma 1.6. *Let V be a vector field over a domain U and suppose that U is the disjoint union of U_1 and U_2. Then if V_1 and V_2 denote V restricted to U_1 and U_2 respectively, we have*

$$\mathrm{Ind}(V) = \mathrm{Ind}(V_1) + \mathrm{Ind}(V_2).$$

□

2. The Index Defined for Compact n-Manifolds

The otopy extension property. *Let V be a continuous vector field on a closed manifold N. Let U be an open set in N. Any continuous proper otopy of V on the domain U can be extended to a continuous homotopy of V on all of N. In fact, if V and W are continuous vector fields with a proper continuous otopy between restrictions of them to open sets, then the otopy can be extended to a continuous homotopy of V to W.*

Proof. The continuous proper otopy implies there is a continuous vector field W on an open set T in $N \times I$ which extends to the closure of T with no zeros on the Frontier and which is V when restricted to $N \times 0$. This vector field W can be thought of as a cross-section to the tangent bundle over $N \times I$ defined over a closed subset. It is well known that cross-sections can be extended from closed sets to continuous cross-sections over the whole manifold. □

We assume that the index is defined for $(n-1)$-manifolds and that all the lemmas of Section 1 hold.

First we consider the case of connected compact manifolds M We suppose that V is a globally defined proper vector field on such a manifold M. We choose a vector field N on the boundary ∂M which points outside of M. Every vector v at a point m on ∂M can be uniquely written as $v = t + kN(m)$ where t is a vector tangent to ∂M and k is some real number. We say t is the *projection* of v tangent to ∂M. Then ∂V is the vector field obtained by projecting V tangent to ∂M. Now we define $\partial_- V$ by restricting ∂V to $\partial_- M$, the set of points such that V is pointing inward. Then we define

$$(*) \qquad \operatorname{Ind}_M(V) = \chi(M) - \operatorname{Ind}_U(\partial_- V) \text{ where } U = \partial_- M.$$

Lemma 2.1. $\operatorname{Ind}_M(V)$ *is well-defined.*

Proof. We assume already defined the index on $(n-1)$-dimensional manifolds with open domains for proper vector fields. Note that $\partial_- V$ is proper on ∂M if V is proper on M, because the Frontier of $\partial_- M$ is a subset of $\partial_0 M$, the subset where V is tangent to ∂M. So a defect of $\partial_- V$ on the Frontier must come from a defect of V on ∂M. Hence $\operatorname{Ind}_U(\partial_- V)$ is defined. Now the vector field $\partial_- V$ obviously depends upon the outward pointing N. If we had another outward pointing vector field N' we would project down to a different $\partial_- V$, call it W. Now the homotopy of vector fields $N_t = tN + (t-1)N'$ always points outside of M for every t. Hence it induces a homotopy from $\partial_- V$ to W and this homotopy is proper. Thus $\operatorname{Ind}(\partial_- V) = \operatorname{Ind}(W)$ by Lemma 1.1. Hence $\operatorname{Ind}_M(V)$ is well-defined for connected manifolds with boundary. If M has empty boundary, then $\operatorname{Ind}_M(V) = \chi(M)$ by $(*)$. Hence $\operatorname{Ind}_M(V)$ is well-defined for all connected manifolds, and hence is well-defined for all N-manifolds. □

REMARK. The above lemma is also true in the case where the normal vector field N is not defined on a closed set of ∂M which is disjoint from the Frontier of $\partial_- M$. Then ∂V is not everywhere defined, but $\partial_- V$ is still proper. A homotopy between N and N', as in the lemma, still induces a proper otopy between $\partial_- V$ and W, so the $\operatorname{Ind}(V)$ is still well-defined in this case also. This case arises when M is embedded as a co-dimension zero manifold in such a way that it has corners. Then the natural outward pointing normal in this situation is not defined on the corners. But we still have the index defined if none of the corners is on the Frontier of $\partial_- M$. This point arises in Theorem 2.6.

Now our goal is to prove that non-zero vector fields have index equal to zero on compact manifolds with boundary.

Theorem 2.2. *On M the globally defined vector field V is properly homotopic to W if and only if*

$$\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- W)$$

for every connected component of ∂M. So as a corollary in the case that ∂M is connected, we have that V is properly homotopic to W if and only if $\mathrm{Ind}(V) = \mathrm{Ind}(W)$.

If V and W are both continuous, then "homotopic" can be replaced by "continuously homotopic" in the statements above.

Proof. We may assume that M is connected. If M has empty boundary, the theorem is true since every globally defined vector field is properly otopic to any other globally defined vector field. So assume that M has non-empty boundary. The theorem is true for manifolds one dimension lower by Lemma 1.1. A proper homotopy of V to W induces a proper otopy from $\partial_- V$ to $\partial_- W$ in the manifold ∂M. Hence $\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- W)$. Hence $\mathrm{Ind}(V) = \mathrm{Ind}(W)$ from $(*)$.

Conversely, we can find a smooth collar $\partial M \times I$ of the boundary so that V restricted to this collar has no defects. Then we homotopy V to V' where V' is defined by $V'(m,t) = tV(m)$ for a point in the collar and $V' = 0$ outside the collar. Now since $\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- W)$ for each connected component of the boundary, we can find a proper otopy from $\partial_- V$ to $\partial_- W$. Now this otopy can be extended to a homotopy of ∂V to ∂W by the otopy extension property. This homotopy in turn can be used to define a proper homotopy from V' to W'. Here we assume W' has the same definition relative to W as V' has to V. Thus W is properly homotopic to V. $\square$

Lemma 2.3. *Suppose V is a proper vector field on a compact manifold M. Let $\partial M \times I$ be a collar of the boundary so small so that V has no defects on the collar. Then V restricted to M minus the open collar $\partial M \times (0, 1]$ has the same index as V.*

Proof. Let ∂V_t denote the projection of V tangent to the submanifold $\partial M \times t$ for every t in I. Let W be the vector field on the collar defined by $W(m,t) = \partial_- V_t$ if (m,t) is a point in $\partial_- M \times t$. Then W can be regarded as a proper otopy, proper since V has no defects on the collar. Thus $\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- V_0)$ and hence $\mathrm{Ind}(V) = \chi(M) - \mathrm{Ind}(\partial_- V)$ equals the index of V restricted to $M' = M - \textit{open collar}$, because the indices of the ∂_- vector fields are the same on their respective boundaries and $\chi(M) = \chi(M')$. $\square$

Lemma 2.4. *Let V be a proper continuous vector field on M. Suppose that $\partial_- V$ is properly otopic to some locally defined vector field W on ∂M. Then there is a proper homotopy of V to a proper continuous vector field X so that $\partial_- X = W$ and the zeros of each stage of the homotopy V_t are equal.*

Proof. Use the otopy extension property to find a homotopy H_t from ∂V to a vector field on ∂M, which we shall call ∂X. Let $n(m,t)$ be a continuous real valued function on $\partial M \times I$ which is positive on the open set T of the otopy between $\partial_- V$ and W, zero on the Frontier of T, and negative in the complement of the closure of T, and so that $n(m,1) = n(m)$ where $V(m) = n(m)N(m) + \partial V(m)$ defines $n(m)$. Such a function exists by the Tietze extension theorem. Using $n(m,t)$, we define a vector field X' on $\partial M \times I$ by $X'(m,t) = n(m,t)N(m) + H_t(m)$. We adjoin the collar to M as an external collar and extend the vector field V by X' to get the continuous vector field X. Now M with the external collar is diffeomorphic to M. Under this diffeomorphism X becomes a vector field which we still denote by X. We may assume this diffeomorphism was so chosen that $X = V$ outside of a small internal collar. Then the homotopy $tX + (1-t)V$ is the required homotopy which does not change the zeros of V. $\square$

Lemma 2.5. *If V is a vector field with no defects on an n-ball B, then* $\mathrm{Ind}_B(V) = 0$.

Proof. For the standard n-ball of radius 1 and center at the origin, we define the homotopy $W_t(r) = V(tr)$. This homotopy introduces no zeros and shows that V is homotopic to the constant vector field. The constant vector field has index equal to zero, as can be seen by using $(*)$. If we have a ball diffeomorphic to the standard ball, then the index of the vector field under the diffeomorphism is preserved by Lemma 1.2, and hence it has the zero index. If the ball is embedded with corners so that the corners are not on the Frontier of the set of inward pointing vectors of V, then the index is defined and by Lemma 2.3 it is equal to the index of V restricted to a smooth ball slightly inside the original ball. This index is zero. $\square$

Theorem 2.6. *If V is a vector field with no defects on a compact manifold M, then* $\mathrm{Ind}_M(V) = 0$.

Proof. Now M can be triangulated and suppose we have proved the theorem for manifolds triangulated by $k-1$ n-simplices. The previous lemma proves the case $k = 1$. We divide M by a manifold L of one lower dimension into manifolds M_1 and M_2 each covered by fewer than k n-simplices so that the theorem holds for them.

We arrange it so that L is orthogonal to ∂M. We use Lemma 2.4 to homotopy V to a vector field with no defects so that the new V is pointing outside orthogonally to ∂M at $L \cap \partial M$. Then a simple counting argument shows that $\mathrm{Ind}_M(V) = 0$ since the restrictions of V to M_1 and M_2 have index zero. This argument works if M has no corners. If M has corners we find a collar of M which gives a smooth embedding of $\partial M \times t$ for all t but the last $t = 1$. Then by Lemma 2.3 above, we find that V, restricted to the manifold bounded by $\partial M \times t$ for t close enough to 1, has the same index as V. That is zero.

The counting argument follows. By induction, $\mathrm{Ind}(V|M_1) = \mathrm{Ind}(V|M_2) = 0$. Thus $\mathrm{Ind}(\partial_- V_1) = \chi(M_1)$ and $\mathrm{Ind}(\partial_- V_2) = \chi(M_2)$. Now we have the following equation $\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- V_1) + \mathrm{Ind}(\partial_- V_2) - \mathrm{Ind}(W)$ where W is the projection of V on the common part of the boundary of M_1 and M_2, that is L. This follows from repeated applications of Lemma 1.6. Now $\mathrm{Ind}(W) = \chi(L)$ since W points outwards at the boundary of L. Hence

$$\mathrm{Ind}(\partial_- V) = \mathrm{Ind}(\partial_- V_1) + \mathrm{Ind}(\partial_- V_2) - \mathrm{Ind}(W) = \chi(M_1) + \chi(M_2) - \chi(L) = \chi(M).$$

Hence $\mathrm{Ind}_M(V) = 0$ from $(*)$. $\square$

3. The Index for Locally Defined Vector Fields

Let N be an n-manifold and let V be a proper vector field on N with domain U. Then the set of defects of V in U is compact. Thus we can find a compact manifold M which contains the defects of V. We define

$$(**) \qquad \mathrm{Ind}_U(V) := \mathrm{Ind}_M(V).$$

Lemma 3.1. $\mathrm{Ind}_U(V)$ *is well-defined.*

Proof. If M and M' are two compact manifolds containing the defects, there is a compact manifold M'' containing both M and M'. The vector field V restricted to $M'' - \mathrm{int}(M)$ is a nowhere zero vector field. Then Theorem 2.6 implies that the index of V restricted to $M'' - \mathrm{int}(M)$ is zero. Now the index of V restricted to M'' equals the index of V restricted to M by the following lemma. □

Lemma 3.2. *Suppose M is the union of two manifolds M_1 and M_2 where the three manifolds are compact manifolds so that the intersection of M_1 and M_2 consist of part of the boundary of M_1 and is disjoint from the boundary of M. Suppose that V is a proper vector field defined on M which has no defects on the boundaries of M_1 and M_2. Then $\mathrm{Ind}_M(V) = \mathrm{Ind}_{M_1}(V_1) + \mathrm{Ind}_{M_2}(V_2)$ where $V_i = V|M_i$.*

Proof.

$$\begin{aligned}\mathrm{Ind}(V) &= \chi(M) - \mathrm{Ind}(\partial_- V)\\ &= \chi(M) - (\mathrm{Ind}(\partial_- V_1) + \mathrm{Ind}(\partial_- V_2) - \mathrm{Ind}(\partial_- V_1|L) - \mathrm{Ind}(\partial_- V_2|L))\end{aligned}$$

by Lemma 1.6 where $L = M_1 \cap M_2$. Now

$$\mathrm{Ind}(\partial_- V_1|L) + \mathrm{Ind}(\partial_- V_2|L) = \mathrm{Ind}(\partial_- V_1|L) + \mathrm{Ind}(\partial_+ V_1) = \chi(L).$$

Thus

$$\mathrm{Ind}(V) = \chi(M_1) + \chi(M_2) - \mathrm{Ind}(\partial_- V_1) - \mathrm{Ind}(\partial_- V_2) = \mathrm{Ind}(V_1) + \mathrm{Ind}(V_2),$$

as was to be proved. □

Lemma 3.3. *Let V be a proper vector field with domain U. Suppose U is the union of two open sets U_1 and U_2 such that the restriction of V to each of them and to $U_1 \cap U_2$ is a proper vector field denoted V_1 and V_2 and V_{12} respectively. Then*

$$(***) \qquad \mathrm{Ind}_U(V) = \mathrm{Ind}_{U_1}(V_1) + \mathrm{Ind}_{U_2}(V_2) - \mathrm{Ind}_{U_{12}}(V_{12}).$$

Proof. We choose disjoint compact manifolds M_1, M_2, and M_{12} containing the zeros of V which lie in $U_1 - U_{12}$ and $U_2 - U_{12}$ and U_{12} respectively. Then the index of V is equal to the index of V restricted to the union of M_1, M_2, and M_{12}. But the index of V_1 is the index of V restricted to M_1 and M_{12}, and the index of V_2 is the index of V restricted to M_2 and M_{12}, and the index of V_{12} is the index of V restricted to M_{12}. Hence counting the index gives the equation $(***)$. □

Corollary 3.4. *The index of a vector field V on a closed manifold M whose domain is the whole of M is equal to $\chi(M)$.*

Proof. This is true by $(*)$ *a priori*. We note that Lemma 3.1 implies that any other way to calculate the index of V will give the same answer. We illustrate, using Lemma 3.2 twice: Let V be a vector field which is non-zero on a small n-ball B about a point. Now let V_1 be V on the n-ball and let V_2 be V on the complement. Then $\mathrm{Ind}(V_1) = 0$, so $\mathrm{Ind}(\partial_- V_1) = 1$. Now $\mathrm{Ind}(\partial_- V_2) = (-1)^{n-1}$. So

$$\mathrm{Ind}(V_2) = \chi(M - B) - (-1)^{n-1} = \chi(M) - (-1)^n - (-1)^{n-1} = \chi(M).$$

Hence $\mathrm{Ind}(V) = \mathrm{Ind}(V_1) + \mathrm{Ind}(V_2) = 0 + \chi(M)$. □

Theorem 3.5. *Given a connected manifold N, two locally defined* (*continuous*) *proper vector fields are properly otopic* (*by a continuous otopy*) *if and only if they have the same index. For every integer n there is a vector field whose index equals that integer* (*provided N has positive dimension*).

Proof. Suppose we have a proper otopy W with domain T on $N \times I$. Let V_t denote W restricted to $N \times t$. We show that there is some interval about t such that V_s has the same index for all s in the interval. Since the set of defects of the otopy is compact we can find a compact manifold M so that $M \times J$, for some closed interval J, lies in T and contains the defects so that the defects avoid $\partial M \times J$. Thus by Theorem 2.2, the proper homotopy V_t on $M \times J$ preserves the index on M, and hence the proper otopy on $N \times J$ preserves the index on N as t runs over J. Thus we have a finite sequence of vector fields each having the same index as the previous vector field. Hence the first and last vector fields have equal indices.

Conversely, for any integer k, let W_k be the locally defined vector field consisting of $|k|$ vector fields defined on disjoint open balls in N, each one of index 1 if $k > 0$ or of index -1 if $k < 0$. Thus $\mathrm{Ind}(W_k) = k$. Now if V has index k, we must show that V is properly otopic to W_k. Now the defects of V form a compact set which is contained in a compact manifold with boundary M so that V is proper and has no defects on the boundary. We may proper otopy V first to a continuous vector field, and then to a smooth vector field. Then we consider V as a cross-section to the tangent bundle of M. Using the transversality theorem, we can smoothly homotopy the cross-section so that it is transversal to the zero section of the tangent bundle keeping the cross-section fixed over the boundary. The dimensions are such that the intersection consists of a finite number of points. Thus we proper otopy V to a vector field with only a finite number of zeros. Now we put small open balls around each of these zeros. The index of the vector field on the ball around each of these zeros is either 1 or -1. Classically this follows from transversality, but we do not need that fact. We may find a diffeomorphic n-ball which contains exactly $|k|$ zeros so that around these zeros the vector field restricts to W_k. The two vector fields have the same index on the n-ball and thus are properly homotopic, since from $(*)$ the index on the boundary of the inward pointing ∂_- vector fields is the same, and so by induction they are properly otopic, hence by the otopy extension property the ∂ vector fields are homotopic. This homotopy can be extended to a homotopy of the two vector fields originally on the n-ball. Then using the sequence of homotopies and otopies, we can piece together a proper otopy of V to W_k. □

REMARK. Note that this proof is more complicated than it need be because it does not use the concept of degree of a map or of intersection number.

Corollary 3.6. *The proper homotopy classes of continuous proper vector fields on a compact manifold with connected non-empty boundary is in one-to-one correspondence with the integers via the index. Of course, the manifold must have dimension greater than one for this to hold.* □

Lemma 3.7. *The index of a locally defined vector field on a manifold N is invariant under diffeomorphism.* □

4. The Index of a Defect

Let V be a vector field on an manifold N. Let D be the set of defects of V. Then D breaks up into a set of connected components D_i. If a component D_i is compact and is an open set in the subspace topology of D, we can define an index denoted $\operatorname{ind}(D_i)$. Note the lower case 'i' here as opposed to the upper case 'I' in the definition of the global and local indices. We call $\operatorname{ind}(D_i)$ the *index of the defect* (*or zero*) D_i.

Definition. If the defect set D is connected, compact and isolated, then we can find a open set U of N containing D and no other defects of V. Then we define the index of D by

$$(****) \qquad \operatorname{ind}(D) := \operatorname{Ind}_U(V).$$

If D is not isolated, then every open set containing D must contain another defect of V. In this case we say $\operatorname{ind}(D) := -\infty$. If D is not compact, we say $\operatorname{ind}(D) := \infty$.

Now if the set of defects of V on N consists of a finite number of compact D_i, then $\operatorname{Ind}_N(V) = \sum_i \operatorname{ind}(D_i)$. However it is possible that V is a proper vector field and there are an infinite number of D_i. Then at least one of the D_i is not isolated in D. But the index of V is still defined. A one dimensional example occurs when M is the interval $[-1, 1]$ and the vector field V is defined by $V(x) = x \sin(1/x)$ for $x \neq 0$ and $V(0) = 0$. Then 0 is a connected component of the defects which is not open in the set of defects of V. Thus $\operatorname{ind}(0) := -\infty$, whereas $\operatorname{Ind}_M(V) = 1$.

If we have an otopy V_t, we imagine the components of the defects D_t as changing under time. We can say that D_{ti} at time t transforms *without topological radiation* into D_{sj} at time s if there is a compact connected component T of the defects of the otopy from time t to time s so that T intersects $N \times t$ in exactly D_{ti} and T intersects $N \times s$ exactly at D_{si}. The index of D_{ti} is the same as the index of D_{sj} if T is compact. In other words if a finite number of "particles" D_i at time t are transformed into a finite number of particles C_j at time s by a compact T, the sum of the indices are conserved.

(1) CONSERVATION LAW.

$$\sum \operatorname{ind}(C_i) = \sum \operatorname{ind}(D_j).$$

Thus the idea of otopy allows us to make precise the concept of defects moving with time and changing with time and undergoing collisions. The index is conserved under these collisions as long as the "world line" T of the component is compact. That is, as long as there are is no "topological radiation", that is as long as the relevant component in the otopy is compact.

As we mentioned in Subsection A of the Introduction, the concept of otopy can be thought of as a space-like vector field in a space time. So if a physicist wants to model something by defects of a vector field, there is a conservation law preserving an invariant of General Relativity which automatically comes along with the model.

5. Properties of the Index

(2) Law of Vector Fields.

$$\mathrm{Ind}(V) + \mathrm{Ind}\ \partial_- V = \chi(M).$$

This is in fact the equation $(*)$ which defines the index. We remark here that any theory of index in which the Law of Vector Fields holds must agree with our definition.

(3) Classification of proper otopy by the index. Let N be a connected manifold. V is properly otopic to W if and only if Ind $V =$ Ind W. If V and W are continuous vector fields, then the otopy can be continuous. For any integer n there is a continuous vector field W so that $n =$ Ind W.

(4) Classification of proper homotopy. Suppose M is a compact connected manifold with non-empty connected boundary ∂M, and suppose V and W are continuous globally defined proper vector fields on M. Then V is properly homotopic to W if and only if Ind $V =$ Ind W. For any integer n there is a continuous proper vector field W so that $n =$ Ind W, provided the dimension of M is greater than one.

In general for M compact, V is proper homotopic to W if and only if Ind $(\partial_- V) =$ Ind $(\partial_- W)$ on every connected component of ∂M.

(5) Poincare-Hopf Theorem. If M is a closed compact manifold and V is a vector field whose domain is all of M, then Ind $V = \chi(M)$.

Proofs. Property (3) is Theorem 3.5. Property (5) is Corollary 3.6. Property (4) follows from Theorem 2.2 and Lemma 1.4, along with the Otopy Extension Property. □

(6) Additivity. Let A and B be open sets and let V be a proper vector field on $A \cup B$ so that $V|A$ and $V|B$ are also proper. Then $\mathrm{Ind}(V|A \cup B) = \mathrm{Ind}(V|A) + \mathrm{Ind}(V|B) - \mathrm{Ind}(V|A \cap B)$.

Proof. Lemma 3.3. □

(7) Existence of defects. If Ind $V \neq 0$ then V has a defect.

Proof. Theorem 2.6 for compact manifolds with boundary. □

(8) Summation equation. Suppose V is a proper vector field and the set of defects consists of a finite number of connected components D_i. Then Ind $V = \sum_i \mathrm{ind}(D_i)$.

Proof. This follows from the definition of $\mathrm{Ind}(D_i)$ and (3). $\square$

(9) Product rule. Let V and W be proper vector fields on A and B respectively. Let $V \times W$ be a vector field on $A \times B$ defined by $V \times W(s,t) = (V(s), W(t))$. Then $\mathrm{Ind}(V \times W) = (\mathrm{Ind}\ V) \cdot (\mathrm{Ind}\ W)$.

Proof. We can assume that A and B are open sets in their respective manifolds. Then V is otopic to V_n where V_n is restricted to a finite set of open sets in A homeomorphic to the interior of J^k when $k =$ dim A and $J = [-1,1]$, so that $V_n(t_1, \ldots, t_k) = (\pm t_1, t_2, \ldots, t_k)$ where the $+t_1$ is taken if Ind V is positive and $-t_1$ is taken if Ind V is negative. The index of the $V_n|J^k$ is ± 1 respectively by (2). So Ind $(V \times W) = \mathrm{Ind}(V_n \times W_n) = \sum\limits_{i,j} \mathrm{Ind}(V_n|J_i^k) \times (W_n|J_j^\ell)$. Now it is easy to see that $\mathrm{Ind}((V_n|J_i^k) \times (W_n|J_j^\ell)) = \mathrm{Ind}(V_n|J_i^k) \cdot \mathrm{Ind}(W_n|J_j^k)$. $\square$

(10) Sign rule.

$$(-1)^n \mathrm{Ind}(V) = \mathrm{Ind}(-V) \quad \text{where } n = \text{ dim } M.$$

Proof. The theorem is true for $n = 1$. Assume it is true for $(n-1)$-manifolds. Now using (2) we have

$$\begin{aligned}
\mathrm{Ind}(-V) &= \chi(M) - \mathrm{Ind}(\partial_-(-V)) \quad \text{by (2)} \\
&= \chi(M) - \mathrm{Ind}(-\partial_+ V) \quad \text{by definition of } \partial_- V \text{ and } \partial_+ V \\
&= \chi(M) - (-1)^{n-1}\mathrm{Ind}(\partial_+(V)) \quad \text{by induction} \\
&= \chi(M) + (-1)^n(\chi(\partial M) - \mathrm{Ind}(\partial_- V))
\end{aligned}$$

since

$$\chi(\partial M) = \mathrm{Ind}(\partial_- V) + \mathrm{Ind}(\partial_+ V).$$

If n is even then

$$\mathrm{Ind}(-V) = \chi(M) + (0 - \mathrm{Ind}(\partial_- V)) = \mathrm{Ind}\ V \quad \text{by (2)}.$$

If n is odd then

$$\begin{aligned}
\mathrm{Ind}(-V) &= \chi(M) - (2\chi(M) - \mathrm{Ind}(\partial_- V)) \\
&= -(\chi(M) - \mathrm{Ind}(\partial_- V)) = -\mathrm{Ind}\ V \quad \text{by (2)}.
\end{aligned}$$

$\square$

(11) Index defines degree. Suppose M is a compact sub-manifold of $\mathbb{R}^n$ of codimension 0. Let $f : M \to \mathbb{R}^n$ be a map so that $f(\partial M)$ does not contain the origin. Define a proper vector field V^f on M by $V^f(m) = f(m)$. Then Ind $V^f =$ *deg* f', where $f' : \partial M \to S^{n-1}$ is given by $f'(m) = \frac{f(m)}{\|f(m)\|}$.

Proof. We homotopy f if necessary so that $\vec{0}$ is a regular value. Then $f^{-1}(\vec{0})$ is a finite set of points. There is a neighborhood of $f^{-1}(0)$ of small balls so that $f : \partial(\text{ball}) \to \mathbb{R}^n - 0 \cong S^{n-1}$. Now, in each of these small balls, f has either degree 1 or -1. If degree equals 1, then $f|\partial(\text{ball})$ is homotopic to the identity. If degree $= -1$, then $f|\partial(\text{ball})$ is homotopic to reflection about the equator. In these cases $\text{Ind}(V^f|\text{ball}) = \pm 1 = deg\ f|\partial(\text{ball})$. Now

$$\begin{aligned}\text{Ind}(V^f) &= \sum \text{ Ind } V^f|(\text{balls}) \qquad \text{by proper otopy} \\ &= \sum deg\ f|\partial(\text{balls}) = deg\ f'.\end{aligned}$$

□

(12) Brouwer Fixed Point Theorem. Suppose $f : M \to \mathbb{R}^n$ where $M \subset \mathbb{R}^n$ is a codimension zero compact manifold. Define $V_f(m) = m - f(m)$. Then Ind $V_f =$ fixed point index of f (assuming no fixed points on ∂M).

Proof. The fixed point index is defined to be the degree of the map $m \to \frac{m-f(m)}{\|m-f(m)\|}$ from $\partial M \to S^{n-1}$. Hence by (11) we have the result. □

(13) Gauss-Bonnet Theorem. Let $f : M \to N$ where M and N are Riemannian manifolds and f is a smooth map. Let V be a vector field on M. Define the pullback vector field $f^*(V)$ by

$$\langle f^*V(m), \vec{v}_m\rangle = \langle V(f(m)), f_*(\vec{v}_m)\rangle.$$

Then if $f : M^n \to \mathbb{R}^n$ so that $f_*|\partial M$ has maximal rank and $f(\partial M)$ contains no zeros of V, then

$$\text{Ind } f^*V = \sum v_i w_i + (\chi(M) - deg\ \hat{N})$$

where $v_i = \text{Ind}(x_i)$ where x_i is the i$^{\text{th}}$ zero of V, w_i is the winding number of $f|\partial M$ about x_i, and $\hat{N} : \partial M \to S^{n-1}$ is the normal (or Gauss) map.

Proof. In paper [G_5]. □

(14) Transfer Theorem. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a smooth fibre bundle with F a compact manifold and B a closed manifold. Let V be a proper vector field on E with vectors tangent to the fibres. Then there is an S-map $\tau : B^+ \to E^+$ so that in ordinary homology $p_* \circ \tau_*$ (cohomology $\tau^* \circ p^*$) is multiplication by the index of V restricted to a fibre, $\text{Ind}(V|F)$.

Proof. In paper [BG]. □

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