FIELDS OF LORENTZ TRANSFORMATIONS ON SPACE-TIME

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Abstract. Fields of Lorentz transformations on a space–time M are related to tangent bundle self isometries. In other words, a gauge transformation with respect to the $-+++$ Minkowski metric on each fibre. Any such isometry $L : T(M) \to T(M)$ can be expressed, at least locally, as $L = e^F$ where $F : T(M) \to T(M)$ is antisymmetric with respect to the metric. We find there is a homotopy obstruction and a differential obstruction for a global F. We completely study the structure of the singularity which is the heart of the differential obstruction and we find it is generated by "null" F which are "orthogonal" to infinitesimal rotations F with specific eigenvalues. We find that the classical electromagnetic field of a moving charged particle is naturally expressed using these ideas. The methods of this paper involve complexifying the F bundle maps which leads to a interesting algebraic situation. We use this not only to state and prove the singularity theorems, but to investigate the interaction of the "generic" and "null" F, and we obtain, as a byproduct of our calculus, a interesting basis for the 4×4 complex matrices, and we also observe that there are two different kinds of two dimensional complex null subspaces.

1. Introduction

Let M be a space–time and let $T(M)$ denote its tangent bundle equipped with the Minkowski inner product $\langle\ ,\ \rangle$ of type $-+++$. A field of Lorentz transformations refers to a vector bundle map $L: T(M) \to T(M)$ inducing the identity $1_M: M \to M$ on M which preserves the inner product. We assume that L preserves orientation and the future light cones.

Now suppose $F: T(M) \to T(M)$ is a bundle map over the identity $1: M \to M$ which is skew symmetric with respect to $\langle\ ,\ \rangle$. That is $\langle Fu, v\rangle = -\langle u, Fv\rangle$. Then the exponential $e^F: T(M) \to T(M)$ is a field of Lorentz transformations, since $e^F = I + F + \frac{F^2}{2} + \dots + \frac{F^n}{n!} + \dots$ and $\langle e^F u, e^F v\rangle = \langle e^{-F} e^F u, v\rangle = \langle e^{-F+F} u, v\rangle = \langle u, v\rangle$. An interesting question, brought to the author's attention by Chris Phillips, is: Under what circumstances is $L = e^F$ for some F?

There is a homotopy obstruction. More interestingly, there is a differential obstruction. We show that the exponential map has a singular point at F if and only if $e^F = I$ (excluding $F = 0$). The kernel of ∇e^F at the singular point is shown to consist of the "orthogonal" component to the two dimensional space of skew symmetric maps which commutes with F.

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These singularities of the exponential map from $\mathfrak{so}(3,1) \to SO(3,1)$ taking $F \mapsto e^F$ are very provocative physically. We may think of a Lorentz transformation as being characterized by an orthonormal frame $\{e_0, e_1, e_2, e_3\}$. As this frame changes under some process, it is difficult to imagine that there is not some 'infinitesimal" process $F(t)$ which yields the motion of the frame by $e^{F(t)}$. Yet if the frame passes through its original position, there is a possibility that there is no covering infinitesimal process.

We might avoid the singularities by introducing two infinitesimal processes $F_1(t)$ and $F_2(t)$ so that $e^{F_1(t)}e^{F_2(t)}$ reproduces the motion of the frame. But, alas, the map $\mathfrak{so}(3,1) \times \mathfrak{so}(3,1) \to SO(3,1)$ also has singular points. If we seek a nice submanifold of $\mathfrak{so}(3,1) \times (\mathfrak{so}(3,1)$ so that the map $(F_1, F_2) \mapsto e^{F_1} \circ e^{F_2}$ induces a proper onto map, there will still be singularities, even though the manifold itself avoids the original singularities. Even if we consider n–processes, we still get singularities.

This is very closely related to the robot arm map, which takes the position of the arm and assigns it to the orientation of the end of the arm. The existence of singularities for an n–linked robot arm follows from the same homotopy argument as the existence of singularities of the Generalized Robot Arm Map which is defined by $\prod^n \mathfrak{so}(3,1) \to SO(3,1)\colon (F_1, \dots, F_n) \mapsto e^{F_1} \dots e^{F_n}$.

When the robot arm approaches a singularity, the parts of the arm moves faster and faster to keep the orientation along its programmed path. At last they drag on each other and the arm departs from the planned path. It could remain on the path only with infinite acceleration.

What happens when a physical process can be described by a moving 4–frame $\{e_0, e_1, e_2, e_3\}$? Does it avoid the singular points somehow; or does it pass through them, in which case, how does it react to the "infinite accelerations" physically? If I could be allowed to speculate, I would say on the basis of the mathematical structure of the singularities, that their existence might give an argument of why spinning objects tend to remain spinning around a fixed or precessing axis, and when this does not obtain, there is an emission or absorption of radiation.

I base my speculation on the following observations. Any $F \in \mathfrak{so}(3,1)$, which is a skew symmetric linear transformation $F\colon \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$, can be classified into three classes, according to its eigenvectors. F has 1, 2 or infinitely many null eigenvector spaces (where a *null vector* s means $\langle s, s\rangle = 0$). The case of infinitely many eigenvectors only occurs for the trivial case, $F = 0$. The case of one null eigenvector subspace corresponds to the null transformations, which the physicists say corresponds to the radiative case in electro–magnetism. The generic case has two null eigenvector spaces.

The singular F are infinitesimal rotations around an axis a multiple of 2π times for some observer. The skew symmetric operators which are "orthogonal" to that F consists of linear combinations of two null operators, one of whose eigenvectors agrees with one of the two eigenvector directions of F, and the other null operator's eigenvector points along the opposite eigenvector of F. Now F plus one of these null operators is still a rotation through an angle $2\pi n$ about an axis, but its observer is different.

Thus we see that a spinning process must maintain the same eigenvectors, hence spin axis, every 2π rotation. And the "directions" having no effect on the spin are given by two null operators.

This paper depends on [Gottlieb (1998)] which may be found on my home page. We keep the same notation. In §2 we will review this notation, but from a generalized complex point of view in which the main features will be stated more quickly. In §3, as an application of this notation, we will produce a basis for the complex 4×4 matrices $M_4(\mathbb{C})$ which are Hermitian, clearly respect the decomposition of $\mathfrak{so}(3,1) \otimes \mathbb{C}$ into two complex quaternionic subalgebras, and in which each of the 16 matrices has a square equal to the identity. This basis easily gives a set of multiplicative generators for the Clifford algebra $Cl(4) \simeq M_4(\mathbb{C})$.

In §4 we will prove the singularity theorem for the exponential map $\mathfrak{so}(3,1) \to SO(3,1)$. We see that the singular points occur at those F such that $e^F = I$ and $F \neq 0$ and the image of the induced differential on the tangent spaces is in a "direction' parallel to F. In §5 we study adjoint mappings, finding several equations relating generic F to their two orthogonal null F. In §6 we discuss the obstructions to representing a Lorentz field by $L = e^F$. In §7, we show that the classical electromagnetic field of a moving charged particle has a very nice description in terms of conjugation by certain exponentials of null F.

I would like to thank the organizers of the conference on The Theory of Fixed Points and its Applications for inviting me to speak last in the cleanup position. From that vantage point I was able to connect some of the material here to concepts arising from various talks which preceded mine. My talk was necessarily philosophical, whereas this paper is completely rigorous, so I welcomed the opportunity to actually quote a precise result or two. In the paragraphs below, I will repeat those connections of this material to fixed point theory.

Lorentz transformations are strongly influenced by their eigenvectors, and of course eigenvectors are particularly useful types of fixed points. In one talk, I believe it was Phil Heath's, an example of a transformation on a two-sphere with only one fixed point was mentioned. A interesting example of those can be defined by a Lorentz transformation with only one eigenvector. Such transformations can be associated to radiation, as mentioned above. The Lorentz transformation takes any future pointing null ray into another future pointing null ray. The future pointing null rays can be characterized as the points on a two-sphere, that is as the direction of a light ray from a given observer.

Robert Brown mentioned a generalization of the Brouwer degree of a map from oriented to unoriented cases. I defined another generalization which works for any continuous map between any topological spaces. I pointed out that I had predicted in print that this generalized concept should appear widely in mathematics, and I mentioned that I had in fact found that this degree determined whether or not a given field of Lorentz transformations had a vector field of null eigenvectors. This can be seen in 7.1 to 7.8 from [Gottlieb (1998)].

2. Notation

We will follow the notation and conventions of [Gottlieb (1998)] as closely as possible. But we shall outline the theory from a different point of view. In [Gottlieb

(1998)], we started with real bundle maps on the tangent space of a space–time M whose tangent bundle was furnished with a Minkowskian metric $\langle\ ,\ \rangle$. There we proceeded as geometrically as possible, defining dot products and cross products with their geometric meanings for example. One main reason for this was to try to understand the role of choices of orientation. As we proceeded, we added more layers of notation, creating at least three different calculi: Level -2 without choice of basis, Level -10 with matrices, and Level -16, the usual (t, x, y, z) of Minkowski space. We found, actually against our wishes, that complexifying was a powerful aid to calculation and understanding.

In this paper, we will begin with complexified Minkowski space $\mathbb{R}^{3,1} \otimes \mathbb{C}$, mostly use Level -10 notation and recover the body of [Gottlieb (1998)] by specializing to the real case and extending to fibre bundles.

Let $\mathbb{R}^{3,1}$ be $\mathbb{R}^4$ with an inner product $\langle\ ,\ \rangle$ of type $-+++$. The space of linear maps $\mathrm{Hom}(\mathbb{R}^{3,1}, \mathbb{R}^{3,1})$ is a 16–dimensional real vector space. Let the 6–dimensional subspace of linear maps F which are skew–symmetric with respect to $\langle\ ,\ \rangle$ be denoted as $\mathfrak{so}(3,1)$. Skew symmetric means $\langle Fu, v\rangle = -\langle u, Fv\rangle$. We say that $u \in \mathbb{R}^{3,1}$ is an *observer* if $\langle u, u\rangle = -1$ and u is future pointing.

We wish to study $\mathbb{R}^{3,1} \otimes \mathbb{C}$. We define the inner product $\langle\ ,\ \rangle_{\mathbb{C}}$ on $\mathbb{R}^{3,1} \otimes \mathbb{C}$ by letting $\langle u, iw\rangle_{\mathbb{C}} = \langle iv, w\rangle_{\mathbb{C}} = i\langle v, w\rangle$ where $v,\ w \in \mathbb{R}^{3,1}$. This extends linearly to an inner product $\langle\ ,\ \rangle_{\mathbb{C}}$ on $\mathbb{R}^{3,1} \otimes \mathbb{C}$. We will usually suppress the $\mathbb{C}$ and write $\langle\ ,\ \rangle$.

Remark 2.1. The usual choice of inner product on $\mathbb{R}^{3,1} \otimes \mathbb{C} \cong \mathbb{C}^4$ is the Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$ which can be defined in terms of $\langle\ ,\ \rangle_{\mathbb{C}}$ by $\langle\langle v, w\rangle\rangle = \langle v, \overline{w}\rangle_{\mathbb{C}}$. The advantage of $\langle\langle\ ,\ \rangle\rangle$ is that it is positive definite on the complex rest space of an observer, there are no null vectors ($s \neq 0$, so that $\langle\langle s, s\rangle\rangle = 0$). The advantage of $\langle\ ,\ \rangle_{\mathbb{C}}$ is that it is linear and there are null vectors! In $\mathbb{R}^{3,1} \otimes \mathbb{C}$, the subspaces of null vectors are two–dimensional and they come in two distinct types. In $\mathbb{R}^{3,1}$, the null spaces are one–dimensional.

Scholium 2.2. In the real case the null spaces are one–dimensional. The propagation of light is along these one–dimensional null spaces. One wonders what the physical meaning of the two types of null spaces is in the complex case? After all, every one who wants to study the real physical world is forced into complexification. God would be malicious if these null spaces had no meaning! I have a suggestion.The two different subspaces correspond to the two different helicities that a photon can have. In other words, one subspace corresponds to a photon moving in the same direction as its Poynting vector, and the other corresponds to a photon moving in the opposite direction as its Poynting vector.

The subspace of $\mathrm{Hom}(\mathbb{C}^4)$ consisting of F which are skew symmetric with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$ is denoted $\mathfrak{so}(3,1) \otimes \mathbb{C}$. Thus $\langle Fv, w\rangle = -\langle v, Fw\rangle$ where $v,\ w \in \mathbb{C}^4$.

The above style of notation comes from Level -2 of [Gottlieb (1998)]. To go to Level -10, we must choose an orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $\mathbb{R}^{3,1}$ so that $\langle e_0, e_0\rangle = -1$ and $\langle e_i, e_j\rangle = \delta_{ij}$ otherwise. These naturally give rise to an orthonormal basis on $\mathbb{R}^{3,1} \otimes \mathbb{C} \cong \mathbb{C}^4$.

This orthonormal basis allows us to write linear transformations $F\colon \mathbb{C}^4 \to \mathbb{C}^4$ as 4×4 matrices, so $\mathrm{Hom}(\mathbb{C}^4) \overset{\cong}{\to} M_4(\mathbb{C})$, the space of 4×4 complex matrices. We see that $\mathfrak{so}(3,1)\otimes\mathbb{C}$ is isomorphic to the vector space consisting of all the matrices of the form

$$\left(\begin{array}{c|ccc} 0 & E_1 & E_2 & E_3 \\ \hline E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{array}\right)$$

We find it convenient to use the block matrix notation

$$\left(\begin{array}{c|c} 0 & \vec{E}^T \\ \hline \vec{E} & \times B \end{array}\right) \text{ where } \vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \text{ and } \times\vec{B} = \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix}.$$

Note that $(\times B)\vec{v} = \vec{v}\times\vec{B}$ if we assume that $\vec{e}_1\times\vec{e}_2 = \vec{e}_3$.

Let us call the space of these matrices S for skew symmetric. Note they are not skew symmetric in the sense of matrices.

Now we introduce the $*$ operation on S. If

$$F = \begin{pmatrix} 0 & E^T \\ E & \times B \end{pmatrix}, \text{ then } F^* = \begin{pmatrix} 0 & -B^T \\ -B & \times E \end{pmatrix}.$$

This $*$ gives rise to a mapping $F\to F^*$ on S to itself which is a linear transformation and $F^{**} = -F$, so the star mapping composed with itself is $-I$ where $I\colon S\to S$ stands for the identity.

The $*$ operation is really a Level -2 concept. It arises from the Hodge dual on two–forms and depends on a choice of a volume form for $\mathbb{R}^{3,1}$. Thus on level -2 we can define $\vec{E}$ and $\vec{B}$ for each observer u. Namely $\vec{E}_u = Fu$ and $\vec{B}_u = -F^*u$ where $\vec{E}_u$ and $\vec{B}_u$ are in the rest space T^u of u, that is $\vec{E}_u$ and $\vec{B}_u$ are orthogonal to u. So $\vec{E}$ and $\vec{B}$ are freed of depending on the whole basis, just the observer.

Now define linear maps $c\colon S\to S$ and $\overline{c}\colon S\to S$ given by $cF\colon= F - iF^*$ and $\overline{c}F\colon= F + iF^*$.

Theorem 2.3. *a) c and $\overline{c}$ are linear maps $\mathfrak{so}(3,1)\otimes\mathbb{C}\to\mathfrak{so}(3,1)\otimes\mathbb{C}$*

b) $c+\overline{c} = 2I$

c) $c\circ\overline{c} = \overline{c}\circ c = 0$

d) $c\circ c = 2c$, $\overline{c}\circ\overline{c} = 2\overline{c}$

Proof.

a)
$$\begin{aligned} c(F+G) &= (F+G) - i(F+G)^* \\ &= F - iF^* + G - iG^* = cF + cG \end{aligned}$$

and $c(aF) = aF - i(aF)^* = aF - i(aF^*) = a(F - iF^*) = acF$.
Similarly for $\overline{c}$.

b) $(c+\overline{c})(F) = F - iF^* + F + iF^* = 2F$.

c)
$$\begin{aligned} c\circ\overline{c}(F) &= c(F+iF^*) = cF + icF^* \\ &= F - iF^* + i(F^* - iF^{**}) \\ &= F - iF^* + iF^* + F^{**} = F + F^{**} = 0. \end{aligned}$$

Similarly for $\overline{c}\circ c = 0$.

d)
$$\begin{aligned} c\circ c(F) &= c(F - iF^*) = cF - icF^* = cF - i(F^* - iF^{**}) \\ &= F - iF^* - iF^* - F^{**} = 2F - 2iF^* = 2cF. \end{aligned}$$

Similarly for $\overline{c}\circ\overline{c}$. $\square$

The above result gives a decomposition of $S = \mathfrak{so}(3,1)\otimes\mathbb{C}$ into the direct sum of two subspaces, $cS\oplus\overline{c}S$. In Level -10 notation, $c\begin{pmatrix} 0 & E \\ E & \times B\end{pmatrix} = \begin{pmatrix} 0 & A \\ A & -i(\times A)\end{pmatrix}$ where $\vec{A} = \vec{E} + i\vec{B}$, and $\overline{c}$ results in $\begin{pmatrix} 0 & \vec{A} \\ \vec{A} & i(\times\vec{A})\end{pmatrix}$ where $\vec{A} = \vec{E} - i\vec{B}$. Note that cS is the $+i$ eigenspace of $*\colon S\otimes\mathbb{C}\to S\otimes\mathbb{C}$ and $\overline{c}S$ is the $-i$ eigenspace of $*$. For $\begin{pmatrix} 0 & \vec{A} \\ \vec{A} & \mp i\times A\end{pmatrix} \overset{*}{\mapsto} \begin{pmatrix} 0 & \pm i\vec{A} \\ \pm iA\vec{A} & \times A\end{pmatrix} = \pm i\begin{pmatrix} 0 & A \\ A & \mp i\times A\end{pmatrix}$.

Now c and $\overline{c}$ restricted to $\mathfrak{so}(3,1)$ are bijections $c\colon\mathfrak{so}(3,1)\xrightarrow{\cong} cS$ and $\overline{c}\colon\mathfrak{so}(3,1)\to\overline{c}S$. So we have two different embedding of $\mathfrak{so}(3,1)$ into $\mathfrak{so}(3,1)\otimes\mathbb{C}$.

Now the beautiful algebra found in [Gottlieb (1998)] follows from the following two facts.

Theorem 2.4. *a) Any element of cS commutes with any element of $\overline{c}S$.*

b) $F\in S$ satisfies $F^2 = kI$ if and only if F is in either cS or $\overline{c}S$.

Proof. Part a) is Theorem 4.8 of [Gottlieb (1998)] and part b) is Theorem 4.5 of the same paper.

Corollary 2.5. *a) $cF\overline{c}G = \overline{c}GcF$ for $F,\ G\in S$.*

b) $cFcG + cGcF = kI$ for some k.

c) $\overline{c}F\overline{c}G + \overline{c}G\overline{c}F = \ell I$ for some ℓ.

Proof. Theorem 4.8 and Theorem 4.7 of [Gottlieb (1998)].

Remark 2.6. The $c\colon\mathfrak{so}(3,1)\to S$ notation was used in [Gottlieb (1998)]. Now if F is real, then $\overline{c}F = \overline{cF}$ and for eigenvalues, $\lambda_{\overline{c}F} = \overline{\lambda}_{cF}$. This is not true in general. This point should be understood in extending the results of §4, §5, §6 to $\mathfrak{so}(3,1)\otimes\mathbb{C}$. The complex conjugate should be restricted to $\overline{c}$. Then most of those results should generalize.

Corollary 2.7. *a)* $cFcG + cFcGF = 2(\vec{A}_1 \cdot \vec{A}_2)I$ *where* $cFu = \vec{A}_1$ *and* $cGu = \vec{A}_2$.

We define $\langle cF, cG\rangle := \langle cFu, cGu\rangle_{\mathbb{C}}$. *This number is independent of the choice of the observer. This is an inner product on* $\mathfrak{so}(3,1) \otimes \mathbb{C}$.

b) $\overline{c}F\overline{c}G + \overline{c}G\overline{c}F = 2(\vec{A}_3 \cdot \vec{A}_4)I$ *where* $\vec{A}_3 = \overline{c}Fu$ *and* $\vec{A}_4 = \overline{c}Gu$.

This defines another inner product $\langle F, G\rangle_- := \langle \overline{c}Fu, \overline{c}Gu\rangle$ *on* $\mathfrak{so}(3,1) \otimes \mathbb{C}$.

We now see from [Gottlieb (1998)] Theorem 4.3 and Corollary 4.4, that $c[F_1, F_2]u = i(cF_1u) \times (cF_2u)$ and $\overline{c}[F_1, F_2]u = -i(\overline{c}F_1u) \times (\overline{c}F_2u)$ and $[cF_1, cF_2]u = 2i(cF_1u) \times (cF_2u)$ with $[\overline{c}F_1,\ \overline{c}F_2]u = -2i(\overline{c}F_1u) \times (\overline{c}F_2u)$.

In general, any $F \in \mathfrak{so}(3,1) \otimes \mathbb{C}$ can be written $F = \frac{1}{2}cF + \frac{1}{2}\overline{c}F$.

$$\begin{aligned}\text{So } [F, G]u &= \frac{1}{4}[(cF + \overline{c}F),\ (cG + \overline{c}G)]u \\ &= \frac{1}{4}\{[cF, cG] + [\overline{c}F, \overline{c}G]\}u = \frac{1}{4}\{2icFu \times cGu + 2(-i)\overline{c}Fu \times \overline{c}Gu\} \\ &= \frac{i}{2}\{cFu \times cGu - \overline{c}Fu \times \overline{c}Gu\}.\end{aligned}$$

This proves:

Theorem 2.8. $[F, G]u = \frac{i}{2}\{cFu \times cGu - \overline{c}Fu \times \overline{c}Gu\}$.

3. An Interesting Basis

We will use our notation to produce an interesting basis for $M_4(\mathbb{C})$, the vector space of complex 4×4 matrices.

We choose an orthogonal coordinate system (t, x, y, z) for Minkowski space $\mathbb{R}^{3,1}$. Thus we are in Level -16 notation of [Gottlieb (1998)]. Let $e_t, \vec{e}_x, \vec{e}_y, \vec{e}_z$ be the unit vectors (with respect to the Minkowski metric). We express $\vec{e}_x = (1,0,0)^T,\ \vec{e}_y = (0,1,0)^T,\ \vec{e}_z = (0,0,1)^T$.

Let

$$E_x := \left(\begin{array}{c|c} 0 & \vec{e}_x^{\,T} \\ \hline \vec{e}_x & 0 \end{array}\right)$$

and define E_y and E_z by replacing $\vec{e}_x$ by $\vec{e}_y$ and $\vec{e}_z$ respectively.

Similarly we define

$$B_x := \left(\begin{array}{c|c} 0 & \vec{0}^T \\ \hline \vec{0} & \times(e_x) \end{array}\right)$$

with B_y and B_z defined by replacing $\vec{e}_x$ by $\vec{e}_y$ and $\vec{e}_z$ respectively.

Lemma 3.1. $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ *form a basis for* $\mathfrak{so}(3,1)$ *as a vector space.*

Now consider $cE_x,\ cE_y,\ cE_z$, and $\overline{c}E_x,\ \overline{c}E_y,\ \overline{c}E_z$. Note that $\overline{c}E_x = \overline{cE_x}$ since E_x is real. Similarly for y and z.

Lemma 3.2. *$\{cE_x,\ cE_y,\ cE_z\}$ forms a basis of the complex vector space $c(\mathfrak{so}(3,1))$ in $\mathfrak{so}(3,1)\otimes\mathbb{C}$. Similarly $\{\overline{cE_x},\ \overline{cE_y},\ \overline{cE_z}\}$ forms a basis for the image of $\overline{c}$.*

Theorem 3.3. *The set of sixteen matrices*

$$\text{a)}\qquad \begin{array}{rrrr} I, & cE_x\overline{cE_x}, & cE_y\overline{cE_y}, & cE_z\overline{cE_z}, \\ cE_x, & \overline{cE_x}, & cE_y\overline{cE_z}, & cE_z\overline{cE_y}, \\ cE_y, & \overline{cE_y}, & cE_x\overline{cE_z}, & cE_z\overline{cE_x}, \\ cE_z, & \overline{cE_z}, & cE_x\overline{cE_y}, & cE_y\overline{cE_x} \end{array}$$

forms a basis for $M_4(\mathbb{C})$, the vector space of 4×4 complex matrices.

b) The square of each of the matrices in the basis is I.

c) Each matrix is Hermitian.

Proof. a) Let e_{ij} represent the 4×4 matrix consisting of a 1 in the i,j position and zeros everywhere else. Then the sixteen e_{ij} form a basis for $M_4(\mathbb{C})$. Express the 16 matrices in a) as linear combinations of the e_{ij} using the order given in a). Then we must show that the coefficient matrix must have linearly independent rows. Although the matrix is 16×16 with 256 entries, most of the entries are zero. In fact, the only rows which can possibly be linearly dependent form four sets. Thus the first four rows are obviously independent of the remaining rows and similarly for the next four rows, and so on. It is easy to show each of the four sets of four rows is linearly independent.

b) Recall that $cF^2 = (\vec{A}\cdot\vec{A})I$ where $\vec{A} = cFe_t = \vec{E}+i\vec{B}$. So $cE_xcE_x = (\vec{e}_x\cdot\vec{e}_x)I = I$. Similarly for $cE_y,\ cE_z$ and $c\overline{E}_x,\ c\overline{E}_y,\ c\overline{E}_z$. Next recall that $cF\overline{c}G = \overline{c}GcF$, so $(c\overline{E}_xc\overline{E}_y)^2 = cE_x^2c\overline{E}_y^2 = I\cdot I = I$. Similar arguments will work for the remaining matrices.

c) Note that the transpose of $\left(\begin{array}{c|c} 0 & \vec{A}^T \\ \hline \vec{A} & \mp i(\times\vec{A}) \end{array}\right)$ is $\left(\begin{array}{c|c} 0 & \vec{A}^T \\ \hline \vec{A} & \pm i(\times\vec{A}) \end{array}\right)$. If $\vec{A}$ is a real vector, then the transpose equals the complex conjugate. Hence

$$\left(\begin{array}{c|c} 0 & \vec{A}^T \\ \hline \vec{A} & \pm i(\times\vec{A}) \end{array}\right) \text{ is Hermitian if } \vec{A} \text{ is real.}$$

Thus $cE_x,\ cE_y,\ cE_z,\ \overline{cE_x},\ \overline{cE_y},\ \overline{cE_z}$ are all Hermitian, so is I. Now $cE_x\overline{cE_y}$ is Hermitian since

$$\begin{aligned}[(cE_x)(\overline{cE_y})]^\dagger &= [(\overline{cE_x})(cE_y)]^T = (cE_y)^T(\overline{cE_x})^T \\ &= (\overline{cE_y})(cE_x) = cE_x\overline{cE_y}.\end{aligned}$$

The remaining matrices are Hermitian by similar arguments. □

We can understand the multiplication properties of these matrices by the following theorem.

Theorem 3.4. *a) $cE_xcE_y = icE_z = cB_z$*

b) $\overline{cE_x}\ \overline{cE_y} = -i\overline{cE_z}$

c) $cE_xcE_y = -cE_ycE_x$ *and* $\overline{cE_x}\ \overline{cE_y} = -\overline{cE_y}\ \overline{cE_x}$

d) $cF\overline{cG} = \overline{cG}cF$

Proof. c) $cE_xcE_y + cE_ycE_x = 2(\vec{e}_x \cdot \vec{e}_y)I = 0$ and apply complex conjugate to this equation.

a) We know that $[cE_x, cE_y]e_t = 2i\vec{e}_x \times \vec{e}_y = 2i\vec{e}_z = 2icE_z$. But since $cE_xcE_y = -cE_ycE_x$ we have $[cE_x, cE_y] = 2cE_xcE_y$. So $(cE_xcE_y)e_t = icE_ze_t$. Now cFe_t completely determines cF. So $cE_xcE_y = cE_z$.

b) Same argument or take complex conjugate of a). $\square$

Thus we see that $cE_x,\ cE_y,\ \overline{cE_x},\ \overline{cE_y}$ generates multiplicitivity $M_4(\mathbb{C})$, and cE_x and cE_y generates the Pauli Algebra, that is, the quaternions complexified.

Now since $M_4(\mathbb{C})$ is the complex Clifford algebra $\mathcal{C}\ell(4)$, there must be generators $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ so that $\alpha_i\alpha_j + \alpha_j\alpha_i = \delta_{ij}I$. One such set of α's is given by

$$\begin{aligned}
\alpha_0 &= cE_x \\
\alpha_1 &= cE_y \\
\alpha_2 &= cE_z\overline{cE_x} \\
\alpha_3 &= cE_z\overline{cE_y}.
\end{aligned}$$

Of course, to obtain γ_i such that $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\langle e_i, e_j\rangle I$ we let $\gamma_0 = i\alpha_0,\ \gamma_i = \alpha_i$ for $i = 1, 2, 3$ or

$$\begin{aligned}
\gamma_0 &= icE_x \\
\gamma_1 &= cE_y \\
\gamma_2 &= cE_z\overline{cE_x} \\
\gamma_3 &= cE_z\overline{cE_y}.
\end{aligned}$$

4. The singularity theorem

Let $\exp\colon \mathfrak{so}(3,1) \to SO(3,1)$ be the exponential map $F \mapsto e^F := I + F + \frac{F^2}{2!} + \frac{F^3}{3!} + \dots$. Our goal is to classify the singularities of exp.

Singularity Theorem. *The only non–regular point of exp is the identity $I \in SO(3,1)$. Any $F \neq 0 \in \mathfrak{so}(3,1)$ satisfies $e^F = I$ if and only if $\lambda_{cF} = 2\pi ni$ where n is a non zero integer. The kernel of* $\exp_*\colon G \mapsto \nabla_G e^F := \frac{d}{dt}(e^{F+tG})|_{t=0}$ *consists of those $G \in \mathfrak{so}(3,1)$ such that $\langle cF, cG\rangle = 0$. That is $\nabla_G e^F = 0$ if and only if $\lambda_{cF} = 2\pi ni \neq 0$ and $\langle cF, cG\rangle = 0$.*

A general theorem for the differential of the exponential map is the following result due to Helgason, see [Helgason (1978)], page 105, Theorem 1.7. This will help simplify our proof of the singularity theorem.

Theorem (Helgason). *Let $\mathfrak{g}$ be the Lie Algebra for a Lie group $\mathbf{G}$. Then* $\nabla_G e^F := \frac{d}{dt}(e^{F+tG})|_{t=0} = e^F \cdot (g[adF](G))$ *where* $adF\colon \mathfrak{g} \to \mathfrak{g}$ *sends* $X \longmapsto [F,X]$ *and* $g[\xi]$ *is the power series of the function* $g(\xi) = \dfrac{e^{-\xi}-1}{-\xi}$.

Corollary 4.1. $\nabla e^F\colon G \mapsto \nabla_G e^F$ *is singular if and only there is a non zero integer* n *so that* $2\pi in$ *is an eigenvalue of* $ad(F)\colon \mathfrak{g} \to \mathfrak{g}$.

Proof. $g(\xi) = 0$ occurs only when $\xi \in 2\pi i(\mathbb{Z} - \{0\})$.

Lemma 4.2. $2c \circ ad(F) = ad(cF) \circ c$.

Proof. We must show that $2c \circ ad(F)(G) = ad(cF)(G)$ for all $G \in \mathfrak{so}(3,1)$. That is that $2c[F,G] = [cF, cG]$. But this last is just Theorem 4.3 of [Gottlieb (1998)].

Lemma 4.3. $\exp\colon \mathfrak{so}(3,1) \to SO(3,1)$ *is singular at* F *exactly when* $\exp\colon S \to M_4(\mathbb{C})$ *is singular at* $\dfrac{1}{2}cF$.

Proof. Suppose $\exp\colon \mathfrak{so}(3,1) \to SO(3,1)$ is singular at F. Then $ad(F)\colon \mathfrak{so}(3,1) \to \mathfrak{so}(3,1)$ has an eigenvalue $2\pi ni \neq 0$. Thus there is an eigenvector $cG \in \mathfrak{so}(3,1) \otimes \mathbb{C}$ so that $ad(F)(cG) = 2\pi nicG$. That is $[F, cG] = 2\pi nicG$. Hence $2\pi nicG = [F, cG] = [\frac{cF}{2} + \frac{\overline{cF}}{2}, cG] = [\frac{cF}{2}, cG]$. Hence $ad(\frac{cF}{2})\colon S \to S$ has the same eigenvalue $2\pi ni$ and hence $\nabla e^{\frac{cF}{2}}$ is singular.

Now $e^{cF} = \cosh(\lambda_{cF})I + \frac{\sinh(\lambda_{cF})}{\lambda_{cF}}\, cF$ according to Theorem 8.5 [Gottlieb (1998)]. Also $e^F = e^{\frac{cF}{2}} e^{\frac{\overline{cF}}{2}}$. So $\exp\colon F \mapsto \frac{cF}{2} \mapsto e^{\frac{cF}{2}} \mapsto e^{\frac{\overline{cF}}{2}} e^{\frac{cF}{2}} = e^F$ is a composition of first an injective map, then an exponential map, and finally an "absolute value" map. Hence the differential ∇e^F factors through the differential $\nabla e^{cF/2}$. Hence if cG is in the kernel of $\nabla e^{cF/2}$, then G must be in the kernel of ∇e^F. That is $\nabla_{cG} e^{cF/2} = 0$ implies $\nabla_G e^F = 0$.

Now we shall show that the kernel of $\nabla e^{cF/2}$ will be 2 complex dimensional in a 3 dimensional complex space. Hence ∇e^F has a kernel of 4 real dimensions. But the subspace of all $G \in \mathfrak{so}(3,1)$ which commutes with F is two dimensional and also it cannot be in the kernel of ∇e^F by the following lemma. Hence the kernel of ∇e^F must be 4 dimensional and so $\nabla_G e^F = 0$ if and only if $\nabla_{cG} e^{cF/2} = 0$. $\square$

Lemma 4.4. *a) If* F *commutes with* $G \neq 0$ *(in any Lie Algebra) then* $\nabla_G e^F \neq 0$.

b) The two dimensional space generated by F *and* F^* *commutes with* F.

Proof. a) Since F commutes with G we have $e^{F+tG} = e^F e^{tG}$. Thus

$$\begin{aligned}\nabla_G e^F &= \frac{d}{dt} e^{F+tG}|_{t=0} = \frac{d}{dt} e^F e^{tG}|_{t=0} = e^F \frac{d}{dt} e^{tG}|_{t=0} \\ &= e^F G \neq 0 \text{ since } e^F \text{ is an isomorphism.}\end{aligned}$$

b) F^* commutes with F by Theorem 3.2 of [Gottlieb (1998)]. $\square$

Thus we will have proved the singularity theorem when we prove the following complex singularity theorem.

Theorem 4.5. *a)* $e^{cF} = I$ *if and only if* $2n\pi i = \lambda_{cF}$ *and* $e^{cF} = -I$ *if and only if* $(2n+1)\pi i = \lambda_{cF}$ *for any integer* n.

b) If $[cF, cG] = 0$, *then* $\nabla_{cG} e^{cF} \neq 0$.

c) Assume $cF \neq 0$ *and* $\lambda_{cF} = n\pi i$. *Then* $\nabla_{cG} e^{cF} = 0$ *if and only if* $\langle cF, cG\rangle = 0$.

Proof. a) From Theorem 8.5 of [Gottlieb (1998)] we know that $e^{cF} = \cosh(\lambda_{cF})I + \dfrac{\sinh(\lambda_{cF})}{\lambda_{cF}}\, cF$. Now $e^{cF} = I$ if and only if $\cosh(\lambda_{cF}) = 1$ and $\dfrac{\sinh(\lambda_{cF})}{\lambda_{cF}} = 0$. If $\lambda_{cF} = 0$, then $e^{cF} = I + cF$, so we can exclude that case. Hence, $\sinh \lambda_{cF} = 0$ if and only if $\lambda_F = n\pi i$. And then $\cosh(n\pi i) = \cos(n\pi) = (-1)^n$.

b) This is Lemma 4.4 a).

c) Let $A_t{:}= cF + tcG$ and let $\lambda_t{:}= \lambda_{A_t}$ be the eigenvalue of A_t.

Then we want to show that $\dfrac{d}{dt}\ \lambda_t|_{t=0} = \dfrac{\langle cF, cG\rangle}{\lambda_{cF}}$.

We know $A_t^2 = \lambda_t^2 I = (\lambda_{cF}^2 + 2t\langle cF, cG\rangle + t^2\lambda_{cG}^2)I$. So $\dfrac{d}{dt}\lambda_t^2 = 2\langle cF, cG\rangle + 2t\lambda_{cG}^2$.

Now formally, $\dfrac{d}{dt}\ \lambda_t^2 = 2\lambda_t \dfrac{d\lambda_t}{dt}$, but caution is needed here, since although λ_t^2 is a well–defined function of t, note that λ_t has an ambiguity of sign and a consistent choice of sign must be made in order to make λ_t a well–defined function of t. Now if λ_t runs over a closed loop in $\mathbb{C}$ around 0, it may not be possible to define λ_t globally. This is no problem in our present argument, but anyone who wants to apply this formula should be careful.

So assuming λ_t is made well–defined, we see that

$$\frac{d}{dt}\lambda_t^2|_{t=0} = 2\lambda_0 \frac{d}{dt}\lambda_t|_{t=0} = 2\langle cF, cG\rangle.$$

Hence

$$\frac{d\lambda_t}{dt}|_{t=0} = \frac{\langle cF, cG\rangle}{\lambda_{cF}}. \tag{*}$$

Now carrying out the differention and using (*) we see that

$$\begin{aligned}\nabla_{cG} e^{cF} &= \frac{d}{dt}(\cosh \lambda_t I + \frac{\sinh \lambda_t}{\lambda_t}(cF + tcG))|_{t=0}\\ &= \alpha I + \beta cF + \gamma cG\end{aligned}$$

where

$$\begin{aligned}\alpha &= \sinh(\lambda_{cF})\frac{\langle cF, cG\rangle}{\lambda_{cF}}\\ \beta &= \frac{\langle cF, cG\rangle}{\lambda_{cF}^3}(\cosh(\lambda_{cF})\lambda_{cF} - \sinh(\lambda_{cF}))\\ \gamma &= \frac{\sinh \lambda_{cF}}{\lambda_{cF}}.\end{aligned}$$

So $\nabla_{cG} e^{cF} = 0$ if and only if $\alpha = \beta = \gamma = 0$. This follows since cF and cG and I are linearly independent since otherwise they would commute Now $\gamma = 0$ if and only if $\lambda_{cF} = n\pi i \neq 0$ and then $\beta = 0$ if and only if $\langle cF, cG\rangle = 0$. $\square$

Corollary 4.6. $\nabla_{cG} e^{cF} =$

$$
\begin{aligned}
&= \langle cF, cG\rangle \left\{ \left(\frac{\sinh(\lambda_{cF})}{\lambda_{cF}}\right) I + \left(\frac{\cosh(\lambda_{cF})}{\lambda_{cF}^2} - \frac{\sinh(\lambda_{cF})}{\lambda_{cF}^3}\right) cF \right\} \\
&+ \frac{\sinh(\lambda_{cF})}{\lambda_{cF}}\, cG \quad \textit{if} \quad \lambda_{cF} \neq 0; \quad \textit{and} \\
\nabla_{cG} e^{cF} &= \langle cF, cG\rangle (I + \frac{1}{3} cF) + cG \quad \textit{if} \quad \lambda_{cF} = 0.
\end{aligned}
$$

Proof. The first equation comes from the above proof. The second equation is the limit of the first equation as cF is chosen so that $\lambda_{cF} \to 0$. L'Hopital's rule is applied to the coefficients.

Corollary 4.7. $\left.\frac{d}{dt}\lambda^2_{cF+tcG}\right|_{t=0} = 0$ *if and only if* $\langle cF, cG\rangle = 0$. $\frac{d}{dt}\lambda^2_{cF+tcG} = 0$ *if and only if* $\langle cF, cG\rangle = 0$ *and* cG *is null.*

5. Exponential equations

Lemma 5.1. *Suppose* F *and* $G \in \mathfrak{so}(3,1)$ *share a null eigenvector* s. *Then* $\langle cF, cG\rangle = \lambda_{cF}\lambda_{cG}$. *In addition, one of* cG *or* cF *is null if and only if* cF *and* cG *anticommute.*

Proof. $(cFcG + cGcF)s = 2\langle cF, cG\rangle s$, hence

$$
\begin{aligned}
2\lambda_{cF}\lambda_{cG} &= 2\langle cF, cG\rangle \\
\lambda_{cF}\lambda_{cG} &= \langle cF, cG\rangle .
\end{aligned}
$$

Now if cF is null, then $\lambda_{cF} = 0$ so $\langle cF, cG\rangle = 0$. So cF and cG anticommute. This argument is reversible. □

Theorem 5.2. *Suppose* $F \in \mathfrak{so}(3,1)$ *and suppose* N *is null and shares a null eigenvector* s *corresponding to* λ_{cF}. *a) Then* $[cF, cN] = 2\lambda_{cF} cN$. *b) If* $N^\dagger$ *is another null operator which shares the other null eigenvector of a generic* cF, *then*

$$[cN, cN^\dagger] = \frac{2\langle cN, cN^\dagger\rangle}{\lambda_{cF}}\, cF.$$

Proof. a) In the case where cF is generic, consider $\frac{1}{\lambda_{cF}} cF$. This has eigenvalues 1 and -1, and so $\frac{1}{\lambda_{cF}} cFu$ is a unit vector. There is a choice of observer u for which this unit vector is a real vector $\vec{k}$ and that $u + \vec{k}$ is parallel to the real eigenvector s.

Now since s is an eigenvector of cN also, we have that $\vec{i} \times \vec{j} = \vec{k}$ where $\vec{i}$ and $\vec{j}$ are parallel to $\vec{E}$ and $\vec{B}$ for cN. That is, $\vec{i}$ and $\vec{j}$ are unit vectors and $cNu = E(\vec{i} + i\vec{j})$ for some constant E. Now $\vec{k} \times (\vec{i} + i\vec{j}) = -i(\vec{i} + i\vec{j})$. So $cFu \times cNu = -i\lambda_{cF} cNu$

when we restore the constants to the above equation. Now $[cF, cN]u = 2i(cFu \times cNu) = 2i(-i\lambda_{cF}cNu) = 2\lambda_{cF}cNu$ by Theorem 4.3 and Corollary 4.4 of [Gottlieb (1998)]. Since the two operators agree on an observer u, they must be equal, so $[cF, cN] = 2\lambda_{cN}cN$.

This relation remains true when cF is null since two null operators are orthogonal if and only if they commute, and $\lambda_{cF} = 0$.

b) Let A and B represent two null elements in cS with null eigenvectors s_+ and s_- respectively. (In order to simplify the equations, we have dropped the c from cA and cB). Let F be a generic element in cS with $Fs_+ = \lambda_F s_+$ and $Fs_- = -\lambda_F s_-$.

Now $AB+BA = 2\langle A, B\rangle I$. Apply this equation to s_+ and s_-, making use of the facts that $As_+ = 0$ and $Bs_- = 0$ and $\dfrac{Fs_\pm}{\lambda_F} = \pm s_\pm$. This gives along with Theorem 5.2 a)

$$[A, B]s_+ = 2\langle A, B\rangle \frac{Fs_+}{\lambda_F}$$

and

$$[A, B]s_- = 2\langle A, B\rangle \frac{Fs_-}{\lambda_F}.$$

Adding these two equations gives

$$[A, B](s_+ + s_-) = 2\langle A, B\rangle \frac{F}{\lambda_F}(s_+ + s_-).$$

Since $s_+ + s_-$ is a time–like vector, this implies that

$$[A, B] = 2\langle A, B\rangle \frac{F}{\lambda_F}\ ,$$

since if two skew operators agree on the same observer they must be the same operator (see Theorem 6.9 of [Gottlieb (1998)] for a vast generalization of this last reason), which is what was desired. □

Corollary 5.3. *$cFcN = \lambda_F cN$ if cN is null and $\langle cF, cG\rangle = 0$.*

Lemma 5.4. *$cGcFcG = 2\langle cF, cG\rangle cG - \lambda_{cG}^2 cF$.*

Proof. Multiply $cFcG + cGcF = 2\langle cF, cG\rangle I$ on the right by cG. □

We will drop the c in cF for ease of notation and say instead that $F \in cS$.

Lemma 5.5. *Let $F, G \in cS$. Then $FG = \langle F, G\rangle I + \frac{1}{2}[F, G]$.*

Proof. $FG = \dfrac{1}{2}\{F, G\} + \dfrac{1}{2}[F, G]$.

Theorem 5.6. *Let $F, G \in cS$. Then $e^F e^G = e^D$ where*

$$\frac{\sinh \lambda_D}{\lambda_D} D = (b\alpha F + a\beta G + \frac{b\beta}{2}\,[F,G]) \text{ and } \cosh \lambda_D = (a\alpha + b\beta\langle F, G\rangle)\ ,$$
$$a = \cosh \lambda_F,\ b = \frac{\sinh \lambda_F}{\lambda_F},\ \alpha = \cosh \lambda_G,\ \beta = \frac{\sinh \lambda_G}{\lambda_G}.$$

We of course assume that $\lambda_D \neq (2n+1)\pi i$

Proof. $e^F e^G = (aI+bF)(\alpha I+\beta G)$, which follows from Theorem 8.5 in [Gottlieb(1998)].

Expand and use the lemma above to get the equations for D and $\cosh \lambda_D$ using again the expansion of e^D from Theorem 8.5 to read off the equations. □

Corollary 5.7. $[e^F, e^G] = \dfrac{\sinh \lambda_F \sinh \lambda_G}{\lambda_F \lambda_G}\ [F,G]$ *if $F, G \in cS$.*

Proof. $[e^F, e^G] = [aI + bF,\ \alpha I + \beta G] = b\beta[F,G]$.

Theorem 5.8. *Let F and G be operators in cS. If $e^F = e^G \neq \pm I$, then $F = G + 2\pi n\hat{B}$ where $\hat{B}$ is unit rotation, that is $\hat{B}$ has eigenvalues i and $-i$. And F and G commute.*

Proof. $0 = [e^F, e^G] = \frac{\sinh \lambda_F}{\lambda_F}\ \frac{\sinh \lambda_G}{\lambda_G}\ [F,G]$. Hence one of λ_F or λ_G must equal πni for $n \neq 0 \in \mathbb{Z}$, in which case $e^F = \pm I$, say; or else $[F,G] = 0$. Hence $I = e^F e^{-G} = e^{F-G}$. Hence $F - G = 2\pi n\hat{B}$ where $\hat{B}$ is a unit rotation.

Corollary 5.9. *If $F, G \in \mathfrak{so}(3,1)$ and $e^F = e^G \neq I$, then $F = G + 2\pi n\hat{B}$ where $\hat{B}$ is unit rotation in $\mathfrak{so}(3,1)$.*

Proof. Now $e^F = e^G$ have the same null eigenvectors, so do F and G. Thus $[F,G] = 0$. Hence $I = e^F e^{-G} = e^{F-G} = e^{2\pi n\hat{B}}$.

Theorem 5.10. *Let $A, B \in cS$. Then $\lambda^2_{[A,B]} = 4(\langle A, B\rangle^2 - \lambda_A^2\lambda_B^2)$.*

Proof. $[A,B]^2 = (AB - BA)(AB - BA)$

$$\begin{aligned}
&= ABAB + BABA - AB^2A - BA^2B \\
&= \{ABA, B\} - 2\lambda_A^2\lambda_B^2 I.
\end{aligned}$$

$$\begin{aligned}
\text{Now } ABA &= -A^2B + 2\langle A,B\rangle A = -\lambda_A^2 B + 2\langle A,B\rangle A \text{ by Lemma 5.4.} \\
\text{So } [A,B]^2 &= \{-\lambda_A^2 B + 2\langle A,B\rangle A, B\} - 2\lambda_A^2\lambda_B^2 I \\
&= -2\lambda_A^2\lambda_B^2 I + 4\langle A,B\rangle^2 I - 2\lambda_A^2\lambda_B^2 I \\
&= 4(\langle A,B\rangle^2 - \lambda_A^2\lambda_B^2)) I.
\end{aligned}$$

□

Corollary 5.11. *If $A, B \in \mathfrak{so}(3,1)$, then*
$$\lambda^2_{c[A,B]} = \langle cA, cB\rangle^2 - \lambda^2_{cA}\lambda^2_{cB}.$$

proof. We know that $2c[F,G] = [cF, cG]$ by Theorem 4.3 of [Gottlieb (1998)]. Hence the eigenvalue squared of $c[F,G]$ is $1/4$ times the eigenvalue squared of $[cF, cG]$. Now apply the preceeding theorem.

Theorem 5.12. *Let A, C be null operators in cS. Then*
$$e^D = e^A e^C = (1 + \langle A, C\rangle)I + A + C + \langle A, C\rangle \hat{E}$$
where $\hat{E}$ is a unit boost operator orthogonal to A and B, and where D is characterized by the vector
$$\frac{\sinh\lambda_D}{\lambda_D}\ \vec{D} := \frac{\sinh\lambda_D}{\lambda_D}\ Du = \vec{A} + \vec{C} + i\vec{A}\times\vec{C}$$

Proof.
$$\begin{aligned} e^A e^C &= (I + A)(I + C) = I + A + C + AC \\ &= I + A + C + \frac{1}{2}\{A, C\} + \frac{1}{2}[A, C] \\ &= I + A + C + \langle A, C\rangle I + \frac{1}{2}(2\langle A, C\rangle\hat{E}) \\ &= (1 + \langle A, C\rangle)I + \langle A, C\rangle\hat{E} + A + C. \end{aligned}$$
The second to last equation follows from Theorem 5.2 b). The characterisation of D follows because
$$[cF, cG]u = 2i\vec{F}\times\vec{G}$$
from [Gottlieb (1998)] Theorem 4.3 and Corollary 4.4 as quoted in the paragraph after Corollary 2.7 in this paper. □

Corollary 5.13. *Let A and C be null operators in cS. Then $e^A = e^C$ if and only if $A = C$.*

Proof. This follows from Theorem 5.8 since $A = C + 2\pi n\hat{B}$, and A and C must commute. Thus $A - C$ is null and equal to a rotation whose eigenvalue is $2\pi ni$. This can only happen if $n = 0$.

Theorem 5.14. *Let $G,\ F \in cS$. Then*
$$\begin{aligned} e^{-G}Fe^G &= ((\cosh\lambda_G)^2 + (\sinh\lambda_G)^2)F \\ &\quad - 2\langle F, G\rangle\frac{(\sinh\lambda_G)^2}{\lambda_G^2}\ G + \frac{(\sinh\lambda_G)(\cosh\lambda_G)}{\lambda_G}\ [F, G]. \end{aligned}$$

Proof. $e^{-G}Fe^G = (aI - bG)F(aI + bG)$.
Expanding yields
$$e^{-G}Fe^G = a^2F + abFG - abGF - b^2GFG.$$
But
$$GFG = -\lambda_G^2F + 2\langle F, G\rangle G.$$
So the result follows.

Corollary 5.15. *Let $F \in cS$ and $N \in cS$ be null and let $\langle F, N\rangle = 0$. Then*

$$e^{-F} N e^{F} = e^{-2\lambda_F} N$$

and

$$e^{-N} F e^{N} = F + 2\lambda_F N.$$

Proof. Apply Theorem 5.2 a) and the fact that $\lambda_N = 0$ to Theorem 5.14 above and simplify using the definition of the hyperbolic trigonometric functions.

Theorem 5.16. *Let $F \in \mathfrak{so}(3,1)$. Let λ_{cF} be imaginary. Then the real exponential satisfies*

$$e^F = \cos^2\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{2}{\lambda_{F^*}^2}\sin^2\left(\frac{\lambda_{F^*}}{2}\right) T_F + \frac{\sin\lambda_{F^*}}{\lambda_{F^*}}\, F.$$

If λ_{cF} is real, then the real exponential is given by

$$e^F = \cosh^2\left(\frac{\lambda_F}{2}\right) I + \frac{2}{\lambda_F^2}\sinh^2\left(\frac{\lambda_F}{2}\right) T_F + \frac{\sinh\lambda_F}{\lambda_F}\, F.$$

If $\lambda_{cF} = 0$, then the real exponential is

$$e^F = I + F + \frac{1}{2}F^2$$

Proof. In the imaginary case we may take $\lambda_{cF} = i\lambda_{F^*}$.

$$\text{So } e^{\frac{cF}{2}} = \cosh\left(i\frac{\lambda_{F^*}}{2}\right) I + \frac{\sinh(i\frac{\lambda_{F^*}}{2})}{i\lambda_{F^*}}\, cF$$
$$= \cos\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{i\sin(\frac{\lambda_{F^*}}{2})}{i\lambda_{F^*}}\, cF$$

$$\text{So } e^{\frac{cF}{2}}\, e^{\frac{\overline{cF}}{2}} = \left(\cos\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{\sin(\frac{\lambda_{F^*}}{2})}{\lambda_{F^*}}\, cF\right)\left(\cos\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{\sin\left(\frac{\lambda_{F^*}}{2}\right)}{\lambda_{F^*}}\, \overline{cF}\right)$$

$$e^F = \cos^2\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{\sin\left(\frac{\lambda_{F^*}}{2}\right)\cos\left(\frac{\lambda_{F^*}}{2}\right)}{\lambda_{F^*}}\,(cF + \overline{cF})$$
$$+ \frac{\sin^2\left(\frac{\lambda_{F^*}}{2}\right)}{\lambda_{F^*}^2}\, cF\overline{cF}$$
$$= \cos^2\left(\frac{\lambda_{F^*}}{2}\right) I + \frac{2\sin\left(\frac{\lambda_{F^*}}{2}\right)\cos\left(\frac{\lambda_{F^*}}{2}\right)}{\lambda_{F^*}}\, F + \frac{2\sin^2\left(\frac{\lambda_{F^*}}{2}\right)}{\lambda_{F^*}^2}\, T_F.$$

Now since $2\sin\alpha\cos\alpha = \sin(2\alpha)$, we obtain the desired result.

The real case and the null case follow similarly. □

Note in the above theorem that $2T_F = F^2$ for null F. Also note that $\left(\frac{T_F}{\lambda_T}\right)^2 = I$, so $\frac{T_F}{\lambda_T}$ is an isometry since it is symmetric with respect to the inner product. So it must be the exponential of something. The following results tells us what that something is.

Theorem 5.17. *Let F be generic in $\mathfrak{so}(3,1)$. Then $T_F = \lambda_T e^{(2n+1)\pi B}$ where $B = -\frac{i}{\lambda_{cF}} \cdot F$ is a rotation. So T_F/λ_{T_F} is a rotation about* $180°$.

Proof. Note that $\lambda_{cB} = -i$. Then

$$e^{(2n+1)\pi B} = e^{(n+\frac{1}{2})\pi cB} e^{(n+\frac{1}{2})\pi \overline{c}B}.$$

Now $e^{(n+\frac{1}{2})\pi cB} = (-1)^n cB$. Hence $e^{(2n+1)\pi B} = ((-1)^n cB)(-1)^n \overline{c}B) = cB\overline{cB}$. Now $cB = -\frac{i}{\lambda_{cF}}F$. So $cB\overline{cB} = cF\overline{cF}/\lambda_{cF}\lambda_{\overline{c}F} = 2T_F/2\lambda_{T_F} = T_F/\lambda_{T_F}$. □

6. Topology and Physics

We look at the lifting problem. Suppose we have a Gauge transformation, that is a field of Lorentz transforms, that is a bundle isometry L

$$\begin{array}{ccc} T(M) & \xrightarrow{L} & T(M) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ M & \xrightarrow{1} & M \end{array}$$

Then we can regard L as a cross–section to the associated bundle $SO(3,1) \to E \xrightarrow{p} M$ where each point e of E is an isometry on the tangent space based on the point $p(e) \in M$. Now suppose $L = e^F$ where $F\colon M \to E'$ is a cross–section where $E' \to M$ is the bundle of tangents along the fibres of E. It has fibres $\mathfrak{so}(3,1)$. Since F can be homotopied to the zero section, it follows that its image e^F is homotopic to the identity I.

A more concrete picture arises when we consider M with trivial tangent bundle. Then the above cross–section L can be represented by a map $f\colon M \to SO(3,1)$. Then there exists an F so that $e^F = L$ only if there is a g which lifts f in the diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & \mathfrak{so}(3,1) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle exp} \\ M & \xrightarrow{f} & SO(3,1). \end{array}$$

This implies that f is homotopic to a constant map since $\mathfrak{so}(3,1)$ is contractible. Thus it is easy to find non homotopy trivial maps which will not give rise to e^F. But if M is contractible, say, this homotopy obstruction does not exist.

If I is in the image of f, then since I is not regular under exp, it is possible that the maps induced on the tangent bundle are not equal,

$$\exp_* \circ g_* \neq f_*\colon T_m(M) \to T_I(\mathfrak{so}(3,1)),$$

since $\exp_* \circ g_*$ is not onto at a critical point, whereas f_* could be onto, for example.

However, there are a few interesting observations to be made:

First, f may be approximated by a smooth mapping f' so that f' avoids I. Since M is 4–dimensional and $SO(3,1)$ is six dimensional, that is easy to do.

Second, if we consider $S \subset M$ a 3 dimensional space–like slice, then $f\colon S \to SO(3,1)$ maps a 3 dimensional space into a 6 dimensional space. We can define self intersection numbers which propagate in space–time.

Now we give an result which is similar to the singularity theorem for robot arms [Gottlieb (1986, 1988)]. The singularities of the robot arm can physically be seen as infinite accelerations of the joints in the arm. The argument below follows that of the robot arm since the map R defined below is closely related to the robot arm map.

Theorem 6.1. *Let $R\colon \mathfrak{so}(3,1)\times\ldots\times\mathfrak{so}(3,1) \to SO(3,1)$ be the map $(F_1,\ldots,F_n) \mapsto e^{F_1}\ldots e^{F_n}$. Suppose $N \subset \mathfrak{so}(3,1)\times\ldots\times\mathfrak{so}(3,1)$ is a submanifold so that $R\colon N \to SO(3,1)$ is onto and proper, i.e. the pre-image of every compact set is compact. Then R must have a singularity.*

Proof. Since R is proper, by a theorem of Ehresman, if R has no singularities then R is a fibre bundle $F \to N \to SO(3,1)$ where the fibre F is a compact closed manifold. Now R is homotopy trivial since R factors through $\prod_n \mathfrak{so}(3,1)$, a contractible space.

Hence the fibre F is homotopy equivalent to $\Omega SO(3,1) \times N$. But $\Omega SO(3,1)$ has non zero cohomology groups in infinitely many dimensions since the connected components are homotopy equivalent of the following series of loopspaces: $\Omega SO(3,1) \cong \Omega SO(3) \simeq \Omega(S^3)$. Since $\Omega(S^3)$ has cohomology in infinitely many dimensions F cannot be compact. $\square$

7. A physical example

A moving particle of charge q moving along a time–like path in space–time has a classical electromagnetic field given as follows. (See [Parrott (1987)], p. 134 for details, for example.)

Suppose the particle p is situated at a point x in Minkowski space–time M. Suppose p has 4–velocity $u \in T_x(M)$, so $\langle u,u\rangle = -1$. Suppose p undergoes an acceleration $\vec{a}$. Now $\vec{a}$ must be orthogonal to u, that is $\vec{a} \in T_x^u$, the rest space of p at x, or equivalently, the subspace of T_x orthogonal to u. Then the electro–magnetic field of p is defined along the future light cone C at x as follows. Any point z on C is given uniquely by $z = ru + r\vec{w}$ where r is a positive number and $\vec{w}$ is a unit vector in the rest space of p at x, namely $\vec{w} \in T_x^u$. Now at z the electric field for observer u is given by $\vec{E} = q\left(\dfrac{\vec{w}}{r^2} - \dfrac{1}{r}\,\vec{a}_\perp\right)$, whereas $\vec{B} = q\vec{a}_\perp \times \vec{w}$ where $\vec{a}_\perp := \vec{a} - (\vec{a}\cdot\vec{w})\vec{w}$ is the orthogonal part of $\vec{a}$ with respect to $\vec{w}$, Hence $\vec{a}_\perp \cdot \vec{w} = 0$.

So if we let $F_{\vec{a}}$ stand for the skew symmetric field on M due to the particle p, we have that $F_{\vec{a}}u = \frac{q}{r^2}E_{\vec{w}} + \frac{q}{r}N_{\vec{a}}$ where $E_{\vec{w}}$ is the skew symmetric operator so that $cE_{\vec{w}}u = \vec{w}$, or equivalently $E_{\vec{w}}$ has an eigenvector along $u + \vec{w}$ associated with eigenvalue $\lambda = 1$ and $B = 0$. Hence $\lambda_{cE_w} = 1$ is the eigenvalue associated with the eigenvector $u + \vec{w}$. And $N_{\vec{a}}$ is the null operator with $\vec{E} = -\vec{a}_\perp$ and $\vec{B} = \vec{a}_\perp \times \vec{w}$ and eigenvector $s = a_\perp^2 u - a_\perp \times (\vec{a}_\perp \times \vec{w}) = a_\perp^2(u + \vec{w})$. So $N_{\vec{a}}$ shares the eigenvector $(u + \vec{w})$ with $E_{\vec{w}}$.

Let $E_{\text{coul}} = \dfrac{q}{r^2}\, E_{\vec{w}}$ represent the boost operator corresponding to the Coulomb field of the particle at rest in its rest frame. Then with the above notation, the following Theorem states the relationship between the Coulomb field at rest, $N_{\vec{a}}$, and the electro–magnetic field of the particle under an acceleration $\vec{a}$ denoted by $F_{\vec{a}}$.

Theorem 7.1. *$F_{\vec{a}}$ is the image of E_{coul} under conjugation by $Exp(rN_{\vec{a}})$. That is*

$$F_{\vec{a}} = e^{-rN_{\vec{a}}} E_{\text{coul}} e^{rN_{\vec{a}}}$$

Proof. Since $N_{\vec{a}}$ and E_{coul} share an eigenvector, $< N_{\vec{a}},\ E_{\text{coul}} >= 0$. Hence we may apply Corollary 5.15 to $cF_{\vec{a}} = cE_{\text{coul}} + \frac{q}{r} cN_{\vec{a}}$. That gives us $e^{-cN} cF e^{cN} = cF + 2\lambda_{cF} cN$ for null cN orthogonal to cF. So setting cN to $\frac{r}{2} cN_{\vec{a}}$ and cF to E_{coul}, we see that $\lambda_{E_{\text{coul}}} = q/r^2$ and so

$$e^{-\frac{r}{2} cN_{\vec{a}}} cE_{\text{coul}} e^{\frac{r}{2} cN_{\vec{a}}} = cE_{\text{coul}} + \frac{q}{r} cN_{\vec{a}} = cF_{\vec{a}}$$

Conjugating the above equation with $e^{\frac{r}{2}\overline{c}N_{\vec{a}}}$ yields

$$e^{-rN_{\vec{a}}} cE_{\text{coul}} e^{rN_{\vec{a}}} = cF_{\vec{a}}$$

and the real part of this equation is the result to be proved. □

References

Daniel H. Gottlieb (1998), *Skew Symmetric Bundle Maps*, Contemporary Mathematics **. 220**, pp. 117 - 141.

Daniel H. Gottlieb (1986), *Robots and fibre bundles*, Bulletin de la Societe Mathematique de Belgique **38**, 219–223.

Daniel H. Gottlieb (1988), *Topology and the Robot Arm*, Applicandae Mathematicae **11**, 117–121.

Helgason (1978), Sigurdur, *Differentiable Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York.

Stephen Parrott (1987), *Relativistic Electrodynamics and Differential Geometry*, Springer-Verlag, New York.